THE SUMMER MEETING IN NEW YORK

The forty-fourth Summer Meeting and Semicentennial Celebration of the Society was held at Columbia University in New York City from Tuesday to Friday, September 6–9, 1938.

The following report deals with the features of the program regularly associated with a Summer Meeting. The special events of the Semicentennial Celebration are only mentioned briefly. A detailed account of the Celebration will appear in the January issue of the Bulletin.

The meeting proved to be a very successful and enjoyable occasion. Approximately seven hundred mathematicians and guests were in attendance. The following four hundred nineteen members of the Society registered during the meeting:

C. R. Adams, H. S. Adams, R. B. Adams, C. F. Adler, R. P. Agnew, E. B. Allen, E. S. Allen, C. B. Allendoerfer, H. F. Archibald, R. C. Archibald, R. G. Archibald, B. M. Armstrong, L. A. Aroian, H. T. R. Aude, W. L. Ayres, H. M. Bacon, D. H. Ballou, Louis Baron, W. D. Baten, Samuel Beatty, E. R. Beckwith, E. M. Beesley, E. T. Bell, P. O. Bell, A. A. Bennett, A. C. Berry, W. James Berry, W. Johnston Berry, E. E. Betz, Garrett Birkhoff, G. D. Birkhoff, Archie Blake, Gertrude Blanch, G. A. Bliss, Henry Blumberg, R. P. Boas, H. W. Bode, Paul Boeder, Samuel Borofsky, M. A. Bowden, J. W. Bower, C. B. Boyer, A. D. Bradley, Richard Brauer, H. E. Bray, R. W. Brink, J. R. Britton, F. L. Brooks, A. B. Brown, G. S. Bruton, H. E. Buchanan, J. A. Bullard, L. H. Bunyan, R. W. Burgess, J. H. Bushey, Jewell H. Bushey, W. E. Byrne, S. S. Cairns, W. D. Cairns, J. W. Calkin, R. H. Cameron, B. H. Camp, C. C. Camp, W. B. Campbell, M. E. Carlen, I. S. Carroll, R. V. Churchill, C. E. Clark, J. A. Clarkson, M. D. Clement, L. W. Cohen, Nancy Cole, J. B. Coleman, W. B. Coleman, J. L. Coolidge, W. C. Coombs, T. F. Cope, A. H. Copeland, L. P. Copeland, Richard Courant, A. P. Cowgill, W. H. H. Cowles, H. S. M. Coxeter, C. H. Currier, H. B. Curry, D. R. Curtiss, D. R. Davis, C. I. Davison, M. M. Day, L. J. Deck, F. F. Decker, R. L. Dietzold, C. E. Dimick, L. L. Dines, Arnold Dresden, J. C. Durand, W. L. Duren, J. E. Eaton, Carolyn Eisele, Churchill Eisenhart, L. P. Eisenhart, J. S. Elston, G. C. Evans, G. W. Evans, Fay Farnum, F. A. Ficken, William Findlay, T. S. Fiske, W. B. Fite, J. H. Fithian, D. A. Flanders, Edward Fleisher, W. W. Flexner, Tomlinson Fort, R. M. Foster, J. S. Frame, Philip Franklin, Bernard Friedman, Orrin Frink, T. C. Fry, H. G. Fudge, A. S. Gale, C. A. Garabedian, J. J. Gergen, B. P. Gill, R. E. Gilman, Jekuthiel Ginsburg, Wallace Givens, Michael Goldberg, Saul Gorn, S. H. Gould, P. H. Graham, M. C. Gray, J. W. Green, Louis Green, H. M. Griffin, C. C. Grove, Laura Guggenbühl, Margaret Gurney, D. W. Hall, F. C. Hall, N. A. Hall, O. H. Hamilton, Harris Hancock, E. H. Hanson, D. C. Harkin, Philip Hartman, G. G. Harvey, T. W. Hatcher, Alan Hazeltine, L. B. Hedge, G. A. Hedlund, E. R. Hedrick, Robert Henderson, Coleman Herpel, Maximilian Herzberger, Fritz Herzog, R. A. Higdon, E. H. C. Hildebrandt, T. H. Hildebrandt, T. F. Holgate, D. L. Holl, M. P. Hollcroft, T. R. Hollcroft, Charles Hopkins, H. M. Hosford, Harold Hotelling, S. E. Hotelling, E. M. Hull, E. V. Huntington, W. A. Hurwitz, W. L. Hutchings, D. H. Hyers, M. H. Ingraham, Dunham Jackson, S. B. Jackson, Nathan Jacobson, C. G. Jaeger, R. L. Jeffery,

E. D. Jenkins, Walter Jennings, S. A. Joffe, R. A. Johnson, B. W. Jones, Harris Jones, H. A. Jordan, L. C. Karpinski, Edward Kasner, E. S. Kennedy, L. S. Kennison, J. R. F. Kent, R. B. Kershner, S. H. Kimball, S. C. Kleene, J. R. Kline, L. A. Knowler, E. R. Kolchin, C. C. Krieger, A. C. Ladner, M. E. Ladue, F. A. LaMotte, H. G. Landau, A. W. Landers, M. K. Landers, A. E. Landry, R. E. Langer, G. A. Larew, C. G. Latimer, V. V. Latshaw, V. S. Lawrence, Solomon Lefschetz, D. D. Leib, B. A. Lengyel, Howard Levi, Madeline Levin, D. C. Lewis, H. G. Lieber, L. R. Lieber, P. H. Linehan, G. H. Ling, Marie Litzinger, W. R. Longley, E. R. Lorch, R. G. Lubben, E. D. McCarthy, N. H. McCoy, R. S. McKee, M. G. McNeil, E. J. McShane, L. A. MacColl, J. K. L. MacDonald, C. C. MacDuffee, Roy MacKay, Saunders MacLane, N. B. MacLean, H. M. MacNeille, H. F. MacNeish, W. G. Madow, Morris Marden, W. T. Martin, R. G. Mason, T. E. Mason, A. E. Meder, G. M. Merriman, A. D. Michal, Norman Miller, G. R. Mirick, E. B. Mode, Virginia Modesitt, E. C. Molina, Deane Montgomery, E. I. Moody, C. N. Moore, Lillian Moore, R. L. Moore, E. M. Morenus, F. M. Morgan, Max Morris, Richard Morris, A. P. Morse, D. S. Morse, Marston Morse, E. J. Moulton, H. T. Muhly, G. W. Mullins, F. J. Murray, S. B. Myers, J. L. Nagle, Tadasi Nakayama, L. I. Neikirk, D. L. Netzorg, R. E. O'Connor, G. A. O'Donnell, Rufus Oldenburger, E. G. Olds, E. R. Ott, F. W. Owens, H. B. Owens, J. C. Oxtoby, Gordon Pall, T. P. Palmer, E. W. Paxson, W. F. Penney, F. W. Perkins, R. M. Peters, H. B. Phillips, A. E. Pitcher, M. H. Plass, Hillel Poritsky, E. L. Post, I. R. Pounder, E. A. Pratt, Walter Prenowitz, G. B. Price, R. G. Putnam, Tibor Radó, G. Y. Rainich, S. E. Rasor, H. W. Raudenbush, E. H. Raybould, G. E. Raynor, M. S. Rees, C. F. Rehberg, W. T. Reid, Haim Reingold, R. W. Rempfer, C. N. Reynolds, C. E. Rhodes, Moses Richardson, R. G. D. Richardson, D. E. Richmond, H. L. Rietz, R. F. Rinehart, John Riordan, J. F. Ritt, H. P. Robertson, G. de B. Robinson, S. L. Robinson, J. B. Rosenbach, S. G. Roth, B. M. Rubin, N. E. Rutt, Arthur Sard, Max Sasuly, A. C. Schaeffer, S. A. Schelkunoff, O. F. G. Schilling, I. J. Schoenberg, G. E. Schweigert, W. T. Scott, Wladimir Seidel, Stephan Serghiesco, W. E. Sewell, I. M. Sheffer, E. W. Sheldon, L. W. Sheridan, Max Shiffman, J. A. Shohat, L. P. Siceloff, L. G. Simons, M. E. Sinclair, James Singer, Abraham Sinkov, H. L. Slobin, M. M. Slotnick, L. L. Smail, M. F. Smiley, A. H. Smith, C. E. Smith, P. A. Smith, Virgil Snyder, F. W. Sohon, E. S. Sokolnikoff, I. S. Sokolnikoff, O. H. Stecker, R. P. Stephens, J. J. Stoker, R. W. Stokes, E. B. Stouffer, E. C. Strayhorn, W. C. Strodt, D. J. Struik, M. M. Sullivan, J. L. Synge, J. D. Tamarkin, A. H. Taub, J. S. Taylor, M. E. Taylor, W. C. Taylor, H. P. Thielman, G. B. Thomas, E. M. Torrance, R. E. Traber, J. I. Tracey, W. J. Trjitzinsky, A. W. Tucker, J. W. Tukey, Annita Tuller, G. B. Van Schaack, R. L. Vitale, M. E. G. Waddell, R. W. Wagner, G. W. Walker, R. J. Walker, J. L. Walsh, R. M. Walter, S. E. Warschawski, J. H. Weaver, Warren Weaver, J. V. Wehausen, Louis Weisner, M. E. Wells, E. T. Welmers, Albert Wertheimer, A. H. Wheeler, C. H. Wheeler, O. L. Wheeler, H. S. White, E. A. Whitman, Hassler Whitney, D. V. Widder, Norbert Wiener, E. P. Wiggin, L. R. Wilcox, R. L. Wilder, S. S. Wilks, C. R. Wilson, E. W. Wilson, R. M. Winger, Aurel Wintner, Audrey Wishard, L. A. Wolf, M. C. Wolf, F. L. Wren, B. F. Yanney, C. H. Yeaton, M. M. Young, Leo Zippin, O. J. Zobel.

One of the special features of this meeting was the series of ten addresses which were delivered at the invitation of the Subcommittee on Invited Speakers of the Committee on the Semicentennial Celebration. Details will be given in the report of the Celebration.

All sectional sessions were held on Tuesday, September 6, in various rooms of the School of Business. At 9:30 A.M. there were four sections: Analysis, Professor R. E. Langer presiding, Professor G. A. Hedlund, secretary; Algebra, Professor J. F. Ritt presiding, Dr. E. R. Lorch, secretary; Geometry and Foundations, Professor A. A. Bennett presiding, Professor G. B. Price, secretary; Statistics and Applied Mathematics, Professor T. H. Hildebrandt presiding, Professor I. J. Schoenberg, secretary. At 2:00 P.M. there were three sections: Analysis, Professor J. L. Walsh presiding, Dr. Nelson Dunford, secretary; Algebra and Theory of Numbers, Professor W. B. Fite presiding, Dr. R. H. Cameron, secretary; Topology and Point Sets, President R. L. Moore presiding, Professor Deane Montgomery, secretary.

Registration headquarters were in the rotunda of Low Library, Columbia University. This spacious room served well both for registration activities and as a place for meeting and conversation.

All desiring dormitory rooms were housed in the Columbia University dormitories, families and women in Johnson Hall and men in John Jay Hall. Meals were served in the Men's Faculty Club of Columbia University.

On Monday afternoon, September 5, in Johnson Hall, the Departments of Mathematics of Columbia University, New York University, Brooklyn College, College of the City of New York, Hunter College, and Queens College entertained the mathematicians and their guests at tea.

On Tuesday evening, Columbia University welcomed the mathematicians and their guests at a reception given in Johnson Hall. Following this, at 9 P.M., the opening session of the Semicentennial Celebration was held in McMillin Theater, President R. L. Moore presiding. The main features of the evening's program were the presentation of delegates from scientific organizations, an address of appreciation to Columbia University by Vice President R. E. Langer, and an address of welcome by Nicholas Murray Butler, President of Columbia University. The founder of the Society, Professor Emeritus Thomas Scott Fiske, of Columbia University, was on the platform.

A panoramic photograph of those attending the meeting was taken at noon, Wednesday, on the steps of Low Library.

On Wednesday, a luncheon for women mathematicians was held at the Faculty Club.

The Semicentennial banquet at the Hotel Astor, Wednesday evening, was attended by three hundred sixty-eight mathematicians and guests.

The exhibitions of the Plimpton, Smith, and Dale collections of manuscripts and books, the models and instruments at Teachers College, and the Astronomical Hollerith-Computing Bureau, were a much appreciated adjunct to the program.

A number of interesting excursions were arranged by the local committee. These included a boat trip on the Hudson to the United States Military Academy at West Point, a visit to the Hayden Planetarium for a special showing, and trips to Jones Beach and to the 1939 World's Fair site.

The Council met on Wednesday, September 7, at 12:45 P.M. in the Faculty Club, and later held an adjourned meeting in the Social Room of the School of Business. The Board of Trustees held a session at 12:45 P.M., Friday, in the Faculty Club. The actions of the Council are given below.

The Secretary announced the election of the following thirteen persons to membership in the Society:

Professor Joseph Barnett, Jr., Oklahoma Agricultural and Mechanical College;
Professor William Allen Barnett, Burlington High School, Burlington, Ky.;
Dr. William Z. Birnbaum, New York University;
Professor William Byron Brown, Mississippi Women's College, Hattiesburg, Miss.;
Mr. Louis Burgess, 17 Battery Place, New York, N. Y.;
Professor Jesse Gerald Chaney, Texas Agricultural and Mechanical College;
Dr. Richard James Duffin, Purdue University;
Miss Muriel Anna Hopwood, Kauai High School, Lihue, Kauai, Hawaii;
Mr. Alfred Welwood Jones, Columbia University;
Mr. Loyal Frank Ollmann, University of Michigan;
Mr. Edmund Addison Pratt, Philadelphia, Pa.;
Mr. Edmund Henry Umberger, University of Maryland;
Professor Fred W. Urban, Central Missouri State Teachers College.

The following appointments by President Moore were reported: as representative of the Society at the Academic Ceremonies on Monday, June 6, in connection with the Centennial Celebration of the first session of Emory and Henry College, Emory, Virginia, Professor W. W. Elliott; as representative of the Society at the concluding Semicentennial Celebration of the Founding of Utah State Agricultural College, Logan, Utah, on June 5–7, Professor V. H. Tingey; as a committee to select the Gibbs Lecturers for 1939 and 1940, Professors A. B. Coble (chairman), G. C. Evans, and M. H. Stone; as a Committee on Resolutions for the Semicentennial meeting, Professors Tomlinson Fort and Richard Morris.

The 1939 spring meeting in Chicago is to be held on April 14–15. An invitation to hold the Summer Meeting of the Society in Seattle at the University of Washington at as early a date as possible was

received with thanks by the Council. The following Committee on Arrangements for the 1939 Summer Meeting in Madison was appointed: Professor M. H. Ingraham (chairman), Dr. Elizabeth S. Sokolnikoff, Professors H. P. Evans, and W. L. Ayres.

The Council recommended to the Society that the By-Laws be amended so as to provide for four instead of three on the Editorial Board of the Bulletin. This recommendation was adopted by the Society at a brief business meeting Wednesday afternoon. In accordance with this action, Article I, Section 1, now reads as follows: "The officers of the Society shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, a Librarian, and four Editorial Committees—one of four members for the *Bulletin*, one of three members for the *Transactions*, one of three members for the *Colloquium Publications*, and one consisting of three representatives of the Society on the Board of Editors of the *American Journal of Mathematics*."

The Committee on the Mathematical Symposium, to be held at Duke University in connection with the Society's meeting, April 7–8, 1939, reported that the following three persons have accepted invitations to give addresses: Professors A. B. Coble, Hermann Weyl, and Norbert Wiener.

The editors of the Transactions were authorized to add one hundred pages to the second volume for 1938.

Professors C. N. Moore (chairman), Saunders MacLane, and G. B. Price were constituted a Committee on Publicity to work experimentally during the next year. It has been generally felt that mathematics does not get its full share of science publicity, but that, on the other hand, it is better to have none than to have sensational stories.

Four committees were authorized, to select hour speakers; one committee each for meetings in the east, midwest, and west, and one for annual and summer meetings.

The Committees on the International Congress of Mathematicians, to be held in Cambridge, Massachusetts, in 1940, held two sessions, at 6:30 P.M., Monday, and 12:45 P.M., Tuesday. They reported to the Council and the Trustees that plans are well along. A preliminary notice of the Congress appeared in the September, 1938, number of the Bulletin.

The titles and cross references to the abstracts of the papers read at the regular sessions follow below. The papers were read in the various sections as follows: papers 1 to 11, Analysis (morning); papers 12 to 17, Algebra; papers 18 to 27, Geometry and Founda-

tions; papers 28 to 35, Statistics and Applied Mathematics; papers 36 to 46, Analysis (afternoon); papers 47 to 55, Algebra and Theory of Numbers; papers 56 to 65, Topology and Point Sets. Papers 66 to 109, whose abstract numbers are followed by the letter *t*, were read by title. Paper 5 was read by Dr. Scott, paper 7 by Dr. Clarkson, paper 9 by Mr. Reade, paper 39 by Professor Moskovitz, paper 40 by Dr. Schaeffer, paper 41 by Professor Sewell, paper 43 by Professor Copeland, paper 48 by Mr. O'Connor, paper 50 by Dr. M. C. Wolf, paper 62 by Professor Montgomery. Professor J. A. Shohat summarized briefly in English the paper presented by Professor Popoff. Of those presenting papers, Dr. Palermo was introduced by Professor L. S. Kennison, Professor Riabouchinsky by Professor R. C. Archibald, Professor Popoff by Dean R. G. D. Richardson, Professor Kamke by Professor A. H. Copeland, Dr. Bergmann by Dean R. G. D. Richardson, Dr. Glenn by Professor T. R. Hollcroft, Dr. Robbins by Professor Hassler Whitney, and Dr. Williams by Professor Walter Bartky.

1. *On singularities and solutions of linear ordinary differential equations*, by J. K. L. MacDonald. (Abstract 44-9-356.)

2. *Asymptotic expansions for certain Bessel functions*, by A. J. Palermo. (Abstract 44-9-366.)

3. *Bounded self-adjoint operators and the problem of moments*, by B. A. Lengyel. (Abstract 44-7-302.)

4. *Sufficient conditions for the convergence of a continued fraction*, by Walter Leighton. (Abstract 44-7-301-*t*.)

5. *A general continued fraction expansion*, by Walter Leighton and W. T. Scott. (Abstract 44-9-351.)

6. *A class of orthogonal functions on plane curves*, by Dunham Jackson. (Abstract 44-9-348.)

7. *The type of certain Borel sets in several Banach spaces*, by C. R. Adams and J. A. Clarkson. (Abstract 44-7-287.)

8. *A solution of the problem of infinite play in chess*, by Marston Morse. (Abstract 44-9-360.)

9. *A characterization of minimal surfaces*, by E. F. Beckenbach and Maxwell Reade. (Abstract 44-7-309.)

10. *Underdetermined systems of linear partial differential equations and Lagrange's multiplier method*, by K. O. Friedrichs. (Abstract 44-9-339-*t*.)

11. *Pseudo-norms in linear spaces*, by D. H. Hyers. (Abstract 44-9-347.)

12. *Ideals in partially ordered sets, ideal extension theorems, and the*

projectivization of incidence geometries (preliminary report), by Saul Gorn. (Abstract 44-5-201.)

13. *Matrix representation of algebraic units and the solution of algebraic equations*, by D. C. Harkin. (Abstract 44-9-343.)

14. *Concerning nil-rings with minimal condition for admissible left ideals*, by Charles Hopkins. (Abstract 44-9-345.)

15. *On symmetric algebras and Galois moduli over modular fields*, by Tadasi Nakayama. (Abstract 44-9-362.)

16. *On decomposition of elements in abelian groups*, by Rufus Oldenburger. (Abstract 44-9-365.)

17. *A method for proving certain abstract groups to be infinite*, by H. S. M. Coxeter. (Abstract 44-9-331.)

18. *Additional properties of the second derivative of a polygenic function*, by J. J. DeCicco. (Abstract 44-7-289.)

19. *Groups of cubes and groups of regular tetrahedra inscriptible in a cube*, by A. H. Wheeler. (Abstract 44-9-383.)

20. *Three-circle problems in modern geometry*, by C. D. Smith. (Abstract 44-9-378-*t*.)

21. *New point configurations and algebraic curves attached to them*, by Arnold Emch. (Abstract 44-9-335-*t*.)

22. *Integral invariants of projective differential geometry*, by P. O. Bell. (Abstract 44-9-327.)

23. *Regular plane curve systems with more cusps than the number possessed by certain irregular systems of the same order*, by T. R. Hollcroft. (Abstract 44-9-344.)

24. *The generalization of Miquel's theorem*, by Henry Gerhardt. (Abstract 44-7-321-*t*.)

25. *Note on a recent set of postulates for the calculus of propositions*, by E. V. Huntington. (Abstract 44-7-316.)

26. *Procedural rules in the propositional calculus*, by T. P. Palmer. (Abstract 44-9-368.)

27. *Galileo's antinomy and the class of numbers defined by their value and origin*, by Dimitri Riabouchinsky. (Abstract 44-9-372.)

28. *Criteria for the reality of apparent periodicities and other regularities* (preliminary report), by Archie Blake. (Abstract 44-7-310.)

29. *The moments of F and of z*, by L. A. Aroian. (Abstract 44-9-324.)

30. *Concerning the distribution of the means of n independent chance variables when each is subject to a certain frequency law*, by W. D. Baten. (Abstract 44-9-326.)

31. *Tubes and spheres in n-spaces, and a class of statistical problems*, by Harold Hotelling. (Abstract 44-9-346.)

32. *Tensor equations equivalent to the Dirac equations*, by A. H. Taub. (Abstract 44-7-306.)

33. *On the subdivision of a mechanical system of corpuscles in equilibrium*, by Stephan Serghiesco. (Abstract 44-9-375.)

34. *Application des méthodes d'intégration de Poincaré à la balistique extérieure*, by Kyrille Popoff. (Abstract 44-9-399.)

35. *On the problem of temperatures in a non-homogeneous bar with discontinuous initial temperatures*, by R. V. Churchill. (Abstract 44-9-329.)

36. *Regularity of function-to-function transformations*, by M. M. Day. (Abstract 44-9-332.)

37. *Superposition on monotonic functions*. II, by Esther M. Torrance. (Abstract 44-9-381.)

38. *The Plateau problem for minimal surfaces which are not relative minima*, by Max Shiffman. (Abstract 44-9-377.)

39. *On convexity in a linear space with an inner product*, by David Moskovitz and L. L. Dines. (Abstract 44-9-361.)

40. *Extension of a theorem of S. Bernstein to non-analytic functions*, by R. J. Duffin and A. C. Schaeffer. (Abstract 44-9-334.)

41. *Note on degree of trigonometric and polynomial approximation to an analytic function*, by J. L. Walsh and W. E. Sewell. (Abstract 44-7-322.)

42. *Some remarks on linear differential systems*, by W. T. Reid. (Abstract 44-9-371.)

43. *A new proof of Sturm's comparison theorems*, by Erich Kamke. (Abstract 44-9-349.)

44. *Theory of non-linear q-difference equations*, by W. J. Trjitzinsky. (Abstract 44-9-382.)

45. *Bicontinuous linear transformations in complex euclidean spaces*, by E. R. Lorch. (Abstract 44-7-293.)

46. *On the theory of harmonic functions of three variables*, by Stefan Bergmann. (Abstract 44-11-435-*t*.)

47. *The number of representations function for binary quadratic forms*, by N. A. Hall. (Abstract 44-7-299.)

48. *The quaternion congruence* $\bar{l}at \equiv b \pmod{g}$, by R. E. O'Connor and Gordon Pall. (Abstract 44-9-364.)

49. *On the location of the roots of real polynomial equations when two roots are equal*, by D. H. Ballou. (Abstract 44-9-325.)

50. *The linear equation in matrices with elements in a division algebra*, by Magarete C. Wolf and Louise A. Wolf. (Abstract 44-9-384.)

51. *On* $\mathfrak{p}$*-adic and modular representations of semi-simple algebras*, by Richard Brauer. (Abstract 44-9-328.)

52. *Relatively highly composite numbers*, by R. G. Archibald. (Abstract 44-9-323.)

53. *The modular covariants* (mod 2) *of formal type, of the quantics of orders less than* 8, by O. E. Glenn. (Abstract 44-9-340.)

54. *Prime and composite polynomials with coefficients in any infinite field*, by Howard Levi. (Abstract 44-9-352.)

55. *Linear diophantine equations in quaternion analysis.* I, by Bernard Friedman. (Abstract 44-9-338.)

56. *On the classification of the mappings of a 2-complex into a space*, by H. E. Robbins. (Abstract 44-9-373.)

57. *Concerning non-alternating interior transformations*, by G. E. Schweigert. (Abstract 44-9-374.)

58. *Induced norms in topological groups*, by E. W. Paxson. (Abstract 44-7-318.)

59. *The extension of homeomorphisms of normal spaces to topologically related spaces*, by R. G. Lubben. (Abstract 44-9-353.)

60. *Some theorems concerning points and continua left invariant by transformations of continua not locally connected*, by O. H. Hamilton. (Abstract 44-7-314.)

61. *On the general point transformation*, by Henry Blumberg. (Abstract 44-9-387.)

62. *Non-abelian compact connected transformation groups of three-space*, by Deane Montgomery and Leo Zippin. (Abstract 44-7-303.)

63. *A class of singular 2-manifolds*, by W. W. Flexner. (Abstract 44-9-336.)

64. *Simultaneous invariants of a complex and subcomplex*, by C. E. Clark. (Abstract 44-9-330.)

65. *Planarity of Peano spaces in terms of homology bases*, by Roy MacKay. (Abstract 44-9-357.)

66. *Asymptotic forms for the generalized Legendre functions*, by G. E. Albert. (Abstract 44-7-288-*t*.)

67. *The significance of the system of subgroups for the structure of the group*, by Reinhold Baer. (Abstract 44-7-308-*t*.)

68. *Additive functionals on lattices*, by Garrett Birkhoff. (Abstract 44-5-282-*t*.)

69. *On Green's functions in the theory of heat conduction*, by H. S. Carslaw and J. C. Jaeger. (Abstract 44-7-319-*t*.)

70. *Asymptotic directions of a field of lineal elements*, by J. J. DeCicco. (Abstract 44-7-290-*t*.)

71. *The derivative clock congruence of a polygenic function*, by Edward Kasner and J. J. DeCicco. (Abstract 44-5-284-*t*.)

72. *Non-commutative arithmetic*, by R. P. Dilworth. (Abstract 44-9-333-*t*.)

73. *Green's function and the problem of Plateau*, by Jesse Douglas. (Abstract 44-7-297-*t*.)

74. *Minimal surfaces of higher topological structure*, by Jesse Douglas. (Abstract 44-7-298-*t*.)

75. *The most general form of the problem of Plateau*, by Jesse Douglas. (Abstract 44-7-311-*t*.)

76. *Remarks on Riemann's doctoral dissertation*, by Jesse Douglas. (Abstract 44-7-312-*t*.)

77. *Totally geodesic Einstein spaces*, by Aaron Fialkow. (Abstract 44-7-313-*t*.)

78. *Duality theorems for singular generalized manifolds*, by W. W. Flexner. (Abstract 44-9-337-*t*.)

79. *Class number relations for the form* x^2-2y^2, by W. H. Gage. (Abstract 44-7-320-*t*.)

80. *Cremona involutions determined by a pencil of quartic surfaces*, by F. C. Gentry. (Abstract 44-7-291-*t*.)

81. *Quaternary Cremona groups of ternary type*, by F. C. Gentry. (Abstract 44-7-292-*t*.)

82. *Twisted cubics associated with a space curve*, by Louis Green. (Abstract 44-9-341-*t*.)

83. *On harmonic separation*, by Archibald Henderson and J. W. Lasley. (Abstract 44-7-315-*t*.)

84. *Polygenic functions whose associated element-to-point transformations T convert the points of the* γ *plane into unions of the z plane*, by Edward Kasner. (Abstract 44-5-283-*t*.)

85. *On complementary manifolds in certain Banach spaces*, by E. R. Kolchin. (Abstract 44-7-300-*t*.)

86. *Semiregular continued cotangents*, by D. H. Lehmer. (Abstract 44-9-350-*t*.)

87. *Generalized regular rings*, by N. H. McCoy. (Abstract 44-7-317-*t*.)

88. *On the degree of convergence of the derivatives of the Birkhoff series*, by W. H. McEwen. (Abstract 44-9-354-*t*.)

89. *A note on Sturm-Liouville expansions and interpolations*, by J. K. L. MacDonald. (Abstract 44-9-355-*t*.)

90. *The structure of automorphism groups of p-adic fields*, by Saunders MacLane. (Abstract 44-9-358-*t*.)

91. *Concerning sets of polynomials orthogonal simultaneously on several ellipses*, by G. M. Merriman. (Abstract 44-9-359-*t*.)

92. *The behavior of a function on its critical set*, by A. P. Morse. (Abstract 44-7-294-*t*.)

93. *The approximate solution of certain linear functional equations*, by E. N. Oberg. (Abstract 44-9-363-*t*.)

94. *On the factorization of generalized quaternions*, by Gordon Pall. (Abstract 44-9-367-*t*.)

95. *On differentiation in Banach spaces*, by B. J. Pettis. (Abstract 44-7-295-*t*.)

96. *Definitions and properties of monotone functions*, by G. B. Price. (Abstract 44-9-369-*t*.)

97. *Some further problems of partitions*, by H. A. Rademacher. (Abstract 44-9-370-*t*.)

98. *An outline of the history and the philosophy of the concept of orientation* (preliminary report), by A. R. Schweitzer. (Abstract 44-7-304-*t*.)

99. *The automorphisms of an abelian group*, by Max Shiffman. (Abstract 44-9-376-*t*.)

100. *Spin representation of conformal groups*, by A. H. Taub. (Abstract 44-7-305-*t*.)

101. *Differential equations and hypercomplex systems*, by Olga Taussky. (Abstract 44-7-296-*t*.)

102. *On factoring a matric polynomial with scalar coefficients*, by H. S. Thurston. (Abstract 44-9-380-*t*.)

103. *On the number of sets conjugate to a matrix with linear elementary divisors*, by H. S. Thurston. (Abstract 44-9-379-*t*.)

104. *Permanent configurations in the problem of five bodies*, by W. L. Williams. (Abstract 44-7-307-*t*.)

105. *On cores of bounded divergent complex sequences and of their transforms by square matrices*, by R. P. Agnew. (Abstract 44-9-385-*t*.)

106. *On algebraic equations whose roots lie in the negative half-plane*, by Hillel Poritsky. (Abstract 44-9-400-*t*.)

107. *A problem on acyclic continuous curves*, by J. H. Roberts. (Abstract 44-9-401-*t*.)

108. *Concerning homeomorphisms of the plane into itself*, by J. H. Roberts. (Abstract 44-9-402-*t*.)

109. *The subgroup of order n of a transitive group of degree n and class n−1*, by Louis Weisner. (Abstract 44-9-404-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE CHICAGO CONFERENCE AND SEMINAR ON ALGEBRA

This seminar and conference were the principal features of the 1938 Summer Quarter program of the Department of Mathematics, University of Chicago, a program intended to emphasize algebra. The seminar began during the week preceding the conference with the following series of foundation lectures: *The structure of algebras* by A. A. Albert; *Lie algebras* by Nathan Jacobson; *Valuations and lattices* by Saunders MacLane; *Introduction to the arithmetics of algebras* by A. A. Albert; *Arithmetic and class-field theory* by Ralph Hull; *Formal power series and polynomial ideals* by Saunders MacLane.

The conference opened at 9:30 A.M., Tuesday, June 27, 1938. H. G. Gale, Dean of the Division of Physical Sciences of the University of Chicago, welcomed the group of 110 in attendance. A. A. Albert spoke on the purposes of the conference. The first lecture was given by L. E. Dickson (Chicago) who traced the history of the integral quantities of algebras. He described the difficulty of obtaining a non-vacuous definition of integral sets of algebras over the rational field and gave his final successful definition. He concluded with a statement of the structure theorems on algebras and the parallel results on their arithmetics. Ralph Hull (Illinois) followed with a description of his recent determination of explicit integral sets of cyclic algebras of prime degree. He also discussed recent work completely determining the class number of normal division algebras of degree greater than two over algebraic number fields, the consequent types of integral sets existing in such algebras, and the theory of their units.

In the afternoon session on Tuesday, A. A. Albert (Chicago) described some of the recent developments in the determination of the structure of normal division algebras, stressing the importance of cyclic algebras. He gave an outline of his recent new proofs of the fundamental properties of cyclic algebras. Richard Brauer (Toronto) then discussed his theory of the factor sets of arbitrary normal division algebras. He showed how these results provide a new method for studying the structure of these algebras, with particular success in the case of algebras of degree five.

The first Wednesday morning lecture was that of Saunders MacLane (Chicago). He formulated the elements of the theory of valuations of fields and discussed his results in connection with the question as to how the structure of a field complete with respect to a discrete valuation is determined by its residue-class field. He also connected the theory of valuations with the two following algebraic-geometric lectures. The first of these, by Oscar Zariski (Johns Hopkins), was on his solution by the use of ideals and valuation theory of the problem of the reduction of the singular points of algebraic surfaces. The second, by Solomon Lefschetz (Princeton), was the only lecture of the afternoon session. He indicated how many of the theorems on algebraic curves, that is on algebraic functions of one variable, which are usually obtained analytically can be obtained more easily by the use of purely algebraic devices. Moreover, the results are then valid for algebraic function fields whose constant coefficient field is other than the complex number field. The discussion of these two papers on applications of algebra was extremely lively.

The Department of Mathematics gave its Summer Quarter dinner Wednesday evening. It was attended by students and faculty as well as those attending the conference. There were 144 present, a group larger than at any previous mathematical dinner at Chicago.

C. C. MacDuffee (Wisconsin) began the Thursday morning session with a discussion of the regular representations of an algebra by matrices. He discussed the correspondence of ideals in a maximal order with certain matrices, and indicated a matrix treatment of the question of the equivalence of ideals. C. G. Latimer (Kentucky) discussed the extensive arithmetic theory of quaternion algebras. He particularly emphasized the connections of ideals with classes of Hermitian forms.

Emil Artin (Notre Dame) opened the afternoon session on Thursday with a discourse on quadratic forms in n variables. He connected the theory with that of the quadratic splitting fields of quaternion algebras, and sketched a new proof of Hasse's p-adic criterion that such a form be a null form. He also gave some results on quadratic forms over a field of characteristic two. Reinhold Baer (North Carolina) followed, speaking on the question as to how much of a group is determined by a given lattice of subgroups. His most complete results were for the case of abelian groups.

The first speaker on Friday morning was John Williamson (Johns Hopkins). He discussed the reduction of normal matrices to canonical form, and showed how the specializations of the type of normality lead to many classical reductions. M. H. Ingraham (Wisconsin) then discussed the matrix equation $BT=TA$ for matrices with elements in a division algebra. This is a generalization of the theory of the similarity of such matrices.

The final session, on Friday afternoon, began with the lecture of O. F. G. Schilling (Johns Hopkins). He showed how many of the theorems on p-adic number fields and their algebraic extensions hold for fields complete with respect to more general valuations. He also applied the theory to a study of normal division algebras over such fields. Nathan Jacobson (North Carolina) then discussed the theory of Lie algebras, in particular simple Lie algebras, over any non-modular field. He told how such algebras may be determined as certain subsets of associative algebras, in particular, of division algebras with an involutorial anti-automorphism.

In the seminar after the conference much of the conference material was discussed in detail in lectures by A. A. Albert and O. F. G. Schilling. In addition Sam Perlis (Chicago) gave his results on the integral sets of rational normal division algebras of composite degree, and Leonard Tornheim (Chicago) gave his work on the integral sets of quaternion algebras over function fields. Saunders MacLane spoke on p-bases of fields of characteristic p and his results on the separability of such fields. D. M. Dribin (National Research Fellow) discussed the class-field theory for non-normal fields. Nathan Jacobson gave the material of his paper on locally compact totally disconnected fields. There was also a long series of lectures by O. F. G. Schilling on generalized algebraic functions.

A list of references and a more complete discussion of some of the material of the conference will be found in Saunders MacLane's paper entitled *Some recent advances in algebra*, which is to be published in the American Mathematical Monthly.

A. A. ALBERT

SHORTER NOTICES

Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques. By S. Stöilow. Paris, Gauthier-Villars, 1938. 10+148 pp.

This volume is another of the well known series, *Collection de Monographies sur la Théorie des Fonctions*, edited by É. Borel. Its purpose is to study the purely topological aspects of the theory of Riemann surfaces and of analytic functions, and to derive some standard theorems and generalizations from this point of view. The key theorem here, that an "interior transformation" is topologically equivalent to an analytic function, was first proved by Stöilow in 1928, in the Annales de l'École Normale, (3), vol. 45, p. 367.

Of course, standard parts of classical function theory are partly topological in nature; we may mention the theorem of Stokes (on which Cauchy's integral theorem is based), the monodromy theorem, and the fundamental theorem of algebra. However, it is not with this side of the subject that Stöilow is concerned, for these theorems cannot be put in purely topological form. The author assumes that the reader is acquainted with classical function theory, including its topological aspects, centering around the Jordan curve theorem.

The first chapter is an introduction to the general theory of topological spaces and of manifolds (particularly 2-dimensional manifolds). Different postulate systems are given, and such topics as the properties of open and closed sets, neighborhoods, compact spaces, and connected sets are studied. It must be said that several definitions and theorems (such as a definition of the term "totalement discontinue," and the theorem that a $(1-1)$ continuous transformation of a compact space is a homeomorphism) are not given, though they are used in later chapters. The second half of the chapter is devoted to the theorem of Brouwer on the invariance of regions. The n-dimensional case is given, using the Sperner proof of the Lebesgue lemma. The author remarks that the 2-dimensional case, which is all that is needed in the book, may be proved much more simply. The average reader will wish that he had given such a proof.

Riemann surfaces are defined, and their relations to analytic functions are given, in the second chapter. Unlike Weyl, Stöilow defines a Riemann surface as being a system composed of a surface V, together with a mapping f of this surface on the (extended) complex plane V_0, certain conditions being satisfied. These conditions are that the interiors of a finite or denumerable set $\delta_1, \delta_2, \cdots$ of closed regions cover V and that, for each i, f be topologically equivalent to $w=z^{n_i}$ for some n_i, in δ_i. That is, there are topological mappings h and h_0 of δ_i and $f(\delta_i)$ into the unit circle in the complex plane such that

$$h_0(f(h^{-1}(z))) = z^{n_i}. \tag{1}$$

(In the definition of Weyl, the function f is not given, but it is assumed that there is an analytic metric given in V; his assumption of the triangulability of V was shown to be unnecessary by Radó.) It is easily seen that the Riemann surface V corresponding to an analytic function f satisfies these conditions. The converse is also proved here, using the method of Weyl, Courant, and Fatou.

The third and fourth chapters are devoted to a study of 2-dimensional manifolds in general, determining what ones can be Riemann surfaces (that is, can be the part V of the pair V, f), and classifying these. To make V a Riemann surface, one must

find a suitable mapping of V on the complex plane. This is possible if V can be covered by a finite or denumerable number of neighborhoods homeomorphic to a region in the plane, and if V is orientable. The classification follows the standard procedure, due to Jordan for closed surfaces, and to Kerékjártó for open surfaces.

In Chapter 5, the topological characterization of analytic functions is given. A transformation f of the space X into the space X_0 is called *equivalent* to the transformation f' of X' into X_0' if there are topological transformations h of X into X' and h_0 of X_0 into X_0' such that

$$f'(p) = h_0(f(h^{-1}(p))).$$

Two properties of mappings which are invariant under this equivalence are the following: The image of any open set is an open set, and no closed connected set containing more than one point goes into a single point. Mappings with these properties are called *interior*. (This term, or *inner*, is now commonly used by topologists to refer to mappings satisfying the first property.) The fundamental theorem is that any analytic function, as a mapping of its Riemann surface on the complex plane, is interior, and conversely, any interior mapping of a surface on the complex plane is equivalent to an analytic function for which the given surface can be taken as its Riemann surface. The essential step in the proof is to show that an interior transformation behaves locally like z^n for some n, as in (1).

The last chapter gives some applications of preceding methods and results, especially to properties of transformations of one Riemann surface into another. The formula of Hurwitz, relating the genus and number of boundaries of each surface to the degree of the transformation and the total amount of branching, is given and generalized. Asymptotic and limiting values of an analytic function are discussed.

The book should prove of real interest to anyone wishing to study deeply into the underlying topological properties of Riemann surfaces and analytic functions. On the whole, the exposition is quite clear, though here and there one finds slight errors and omissions of important details.

HASSLER WHITNEY

Opérations Infinitésimales Linéaires. By Vito Volterra and Bohuslav Hostinsky. Paris' Gauthiers-Villars, 1938. 7+238 pp.

The infinitesimal calculus of linear operators was invented by Volterra in 1887, and it is with unusual interest that one opens a volume written fifty years later on this important subject, when one discovers that he is a coauthor.

Consider a linear operator X which is a function $X(t)$ of the time t. If multiplication is taken as the fundamental operation,* then analogy with ordinary functions suggests letting the quotient $X(t+\Delta t)X^{-1}(t)$ measure the "change" in X during the interval from t to $t+\Delta t$, and

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} [X(t + \Delta t)X^{-1}(t)] \tag{1}$$

measure the "rate of change," or "derivative" (more properly, right-derivative) of $X(t)$. It is natural to regard this derivative as a sort of "infinitesimal linear operator," whence the title of the book.

* If addition is taken as the fundamental operation, one gets the (commutative) infinitesimal calculus of vectors, which was discussed by H. Grassmann in 1862, in his *Ausdehnungslehre*, part 2, chaps. 2–4.

Having defined derivatives, one naturally defines an indefinite "product integral" of $X(t)$ (in symbols, $\int X(t)dt$) as any function whose derivative exists and is $X(t)$. This definition contains as a special case the usual definition of a solution (or integral) of a system of first-order ordinary linear homogeneous differential equations.

Moreover, the familiar theorem that if $Y(t)$ is one solution then the general solution is $Y(t)C$—obtained by right-multiplying $Y(t)$ by a constant matrix—becomes identified with the theorem that if $y(t)$ is one integral of $x(t)$, then the general integral is $y(t)+C$. Likewise, the proof of the existence of integrals of such differential equations becomes merged with Riemann's constructive proof of the existence of a "definite integral" for all continuous functions.

The above ideas were contained in Volterra's original paper; together with some standard material on the canonical form of (complex) matrices, they suggest the content of Chapters I–VI of *Opérations Infinitésimales Linéaires*.

Chapters VII–XI are devoted to new, but closely related topics: partial differentiation and path integration of functions of several real variables, whose values are linear operators. A necessary and sufficient condition that a path integrand be a total differential is found—this is the usual condition of "complete integrability," and gives a fresh interpretation to the latter. The study of partial derivatives also paves the way for the study of analytic matrix functions $X(z)$ of a complex variable; analogies exist to most parts of the usual theory, including contour integrals, for example.

Chapters XII–XV are consecrated to Volterra's early (1895) applications of this last theory to ordinary complex functions. The monodromie group is shown to be composed of the analogs of residues, and "Fuchs' results on linear differential equations are shown to be the extension of Cauchy's theorem on residues." The many other applications include exhaustive classifications of the singularities of the integrals of linear differential equations of various types.

In the last three chapters (XVI–XVIII), Hostinsky tries to apply Volterra's methods to linear operators on function spaces, and indirectly to the theories of heat and diffusion. These are the most provocative chapters of the book.

In the theory of heat, if $f(x)$ expresses the temperature of a bar of unit length as a function of position at time s, then the temperature at a later time t is a function $g(x)$ obtainable by applying to $f(x)$ a (linear) "integral operator of the first kind,"

$$G(s, t): \quad f(x) \to \int_0^1 G(x, y; s, t)f(y)dy, \tag{2}$$

where $G(x, y; s, t)$ is a Green's function independent of $f(x)$. A similar situation exists in the theories of diffusion and of dependent probabilities (alias stochastic processes).

The linear operator $G(s, t)$ is well known to be the *integral* of Fourier's (linear) differential operator $\partial^2/\partial x^2$, which is conversely the *derivative* of $G(s, t)$. But unhappily, Volterra's definitions do not include this case; for example, $G(s, t)$ has no inverse (Maxwell, *Theory of Heat*, 1872, p. 241), and so (1) is meaningless.

On the other hand, "integral operators of the second kind": $f(x) \to f(x) + \int_0^1 K(x, y)f(y)dy$, can be handled by Volterra's methods. One can define inverses by Neumann's series $(1+X)^{-1} = 1-X+X^2-X^3+\cdots$, and use (1), provided $|K(x, y)| < \frac{1}{4}$. This yields a limited "infinitesimal calculus of infinite matrices," which includes what Khintchine calls the "Poisson case" of dependent probabilities.

Hostinsky's theory of integral operators of the first kind (Chapter XVIII), on the other hand, bears little resemblance to Volterra's theory. It begins with the identity "of Chapman-Kolmogoroff": $G(s, t)G(t, u) = G(s, u)$ if $s < t < u$. Hostinsky points out that this condition expresses the fact that $G(s, t)$ is the definite integral of

an "infinitesimal linear operator." Kolmogoroff (Mathematische Annalen, vols. 104, 108) had previously, but in less suggestive language, correlated such families of operators with infinitesimal differential operators; technically, the reviewer prefers Kolmogoroff's treatment.

The reviewer also feels that Hostinsky could have used with profit the modern theory of function spaces (von Neumann, Stone, Banach). Again, von Neumann's theory of groups of linear transformations, and Schlesinger's Lebesgue integration of matrices, might well have been sketched—and a reference to Delsarte's work is really called for.

But these defects do not prevent the book from being of the first importance—in fact, probably the best available introduction to the higher theory of linear differential equations.

GARRETT BIRKHOFF

Économique Rationnelle. By Guillaume and Ed. Guillaume. (Actualités Scientifiques et Industrielles, nos. 504–508.) Paris, Hermann, 1937. 375 pp.

An Econometric Approach to Business Cycle Problems. By J. Tinbergen. (Actualités Scientifiques et Industrielles, no. 525.) Paris, Hermann, 1937. 73 pp.

The great increase of the last few years in statistical material relating to the phenomena of economics has brought realization that the "laws of economics" must be subjected to new scrutiny. In some cases they must be modified, in others reformulated with the introduction of new variables. In this modern adventure, to which the name of econometrics has been generically given, mathematics plays its customary central role. Although the origin of mathematical economics antedates the publication in 1838 of A. A. Cournot's classic *Theory of Riches*, which ushered in the justly celebrated works of Jevons, Walras, Marshall, Edgeworth, Pareto, and others of the mathematical school, econometrics in the modern sense is a new science. The books before us for review indicate the trend that modern studies are taking.

The first volume is divided into five parts: (1) Method, (2) Pure economy, (3) Pure economy in interference with the legal domain, (4) The national legal-economic domain in interference with external society, (5) Mathematical models of the economic world.

It is the last part which will be of the most interest to mathematicians, since the authors attempt to establish mathematical models for the interpretation of economic phenomena. Their point of view is taken from dynamics. The authors set up equations which represent for them "the principle of the conservation of the flux of commodities." They introduce the notion of "money liquidity" in a theory of "kinetic and potential money." Economic equilibrium conditions are formulated in terms of the differential variation in commodities. Intriguing new units are employed such as the gold-gram as a quantity of value, the gold-gram/sec. as the flux of value, the gold-gram/man-sec. as the density of flux.

The fundamental tenet of the authors is found in the statement: "We shall see that in attempting to create a *rational economics* similar to a body of doctrine such as rational mechanics, we are led similarly to search for some principle of conservation, more particularly the axiom which we shall call the 'principle of the conservation of value.' "

The prefatory material is elegantly expressed and contains much of interest to the philosophy of science. Bridgman's operational theory is regarded as an essential viewpoint for the new methods. Thus, in criticising the subjective theories of the older economists, the authors remark: "Unfortunately, their efforts have remained

practically sterile, because of a fundamental difficulty which has not yet been resolved: their *axiomatique* was not 'operation.' "

The work of Tinbergen is an attempt to test statistically a mechanism of business cycle policy, which has been developed over a period of years by the author and a group of Dutch mathematical economists. The analysis considers an array of twenty-two variables, and nine "international quantities considered as data." The variables are classified under three heads: (1) prices, such as wage rates, cost of living; (2) physical quantities, such as total employment, total output; (3) value figures, such as total wages' bill, value of exports. These variables are related by twenty-two equations which "are partly of a definitional or technical nature and partly the reflection of 'direct causal relationships.' "

The monograph is devoted to a study of problems which can be formulated in terms of the mathematical system. These problems are of four kinds: (1) to determine the movements defined by the variables for a set of historically given initial values; (2) to extrapolate the natural tendencies of the system, all conditions remaining unchanged; (3) to determine the movements after a variation has been imposed in the system by some "given policy"; (4) to find the optimum variation, that is to say, to determine the "best policy."

Tinbergen's study will be of interest to those who wish to view the full complexity of the economics problem and to see one of the most heroic attempts yet made to reduce the interacting variables to a mathematical system.

H. T. DAVIS

Grundlagen und Methoden der Periodenforschung. By Karl Stumpff. Berlin, Springer, 1937. 332 pp.

As the title indicates, this book deals with the basic ideas of determining the periodic properties of functions and sets of points, and the methods of applying these ideas to empirical functions and observed data. The author, well versed in such analysis from his meteorological work, has taken pains to explain the fundamentals of the subject in a thorough and readily understood manner. Then he goes on to explain in some detail how the fundamentals are best applied to empirical functions obtained from a recording instrument or a series of observations of some physical phenomena.

The problem of expressing a given function as a linear combination of a series of arbitrary functions is attacked by the method of least squares. The equations for the coefficients are derived in general, and the simplified formulas for the case of orthogonal function systems are set down for ready reference. Various function systems are then discussed. These consist of circular functions (which lead to Fourier series expansions), Legendre polynomials, Hermitian polynomials, and Laguerre polynomials.

As a first application, the methods are applied to the smoothing and interpolation of a series of observations.

Next, the technically very important case of the harmonic analysis of observations with one and two independent variables is treated, including several practical schemes for carrying out the detailed calculations.

An analysis is made of the possibility and practicability of expressing an empirical function in the form of a spectrum. The properties and methods of calculating the spectrum of a series of observations is presented.

A chapter on the application of statistical methods to the analysis of the periodic properties of empirical functions serves to introduce distribution functions, correla-

tions, and a broader point of view than the preceding analytical method of least squares.

Other analytical methods for period analysis are considered. The Laplace transformation, differential and difference relations, and the method of exhaustions are included.

Finally, a discussion of physical aids to period analysis—including physical concepts such as momentum, resonance, and interference; and mechanical and optical devices—serves to complete the discussion of methods of application to practical problems.

There is collected in this book a sound treatment of the basis of the period analysis of empirical functions together with detailed discussions of methods of applying these in practice.

Anyone engaged in such period analysis or interested in a sound discussion of fundamentals will find this book very useful.

HOWARD EMMONS

The Axiomatic Method in Biology. By J. H. Woodger. Cambridge, University Press, 1937. 10+174 pp.

We have here the first attempt to build a system of biology on the basis of abstract logic. The book will probably be harder reading than the author (reader in biology at the University of London) realizes—save for those few who are versed both in Russell's symbolism and in fundamental biology. Nonetheless, its writing was a task well worth doing, and one which has been done excellently. It discusses biology with precision of statement and reliability of reasoning, and clearly shows the conceptual unity underlying a number of basic branches. It emphasizes the wisdom of R. A. Fisher's remark: "I can imagine no more beneficial change in scientific education than that which would allow each (mathematician and biologist) to appreciate something of the imaginative grandeur of the realms of thought explored by the other."*

In justification of the undertaking, Woodger quotes A. N. Whitehead: "There are an indefinite number of purely abstract sciences, with their laws, their regularities, and their complexities or theorems—all as yet undeveloped." And in the preface, which calls the book an experiment, he says: "In every growing science there is always a comparatively stable, tidy, clear part, and a growing, untidy, confused part. I conceive the business of theoretical science to be to extend the realm of the tidy and systematic by the application of the methods of the exact or formal sciences, i.e. pure mathematics and logistic."

Almost the whole substance of the book is given in the symbolic statement of axioms, definitions, and theorems. As a concession to the laity, most statements are also explained, more at length, in words. Some 250 symbols are used, mainly taken from the *Principia Mathematica* or introduced for their biological usefulness; the list of these symbols constitutes, in fact, the only index.

The ten undefined signs can be interpreted as "part of," "before in time," "organized unities," "related by a succession of divisions and fusions," "cell," "male," "female," "whole organisms," "environment," "genetic properties." The most important derived concept is that of a "hierarchy," a relation (xRy) which is one-many and asymmetrical, has just one beginner and, as possible terms y, those to which this

* R. A. Fisher, *The Genetical Theory of Natural Selection*, p. ix.

beginning stands in the relation R or a power of R. Thus we might have a species, a man, a cell, or a gene within a cell, together with all its descendants.

More general is the "*dend*," which allows both many-one and one-many relations, so that there need be neither a unique beginner nor a unique final term. The noun "*δένδρον*" is of course the source. In particular we have "*zgdend*" and "*cpdend*"—dendra, respectively, of cells (including gametes which, by fusion, yield zygotes) and of cell-parts (in particular such "continuous" or essential parts as chromosomes and genes). Cytology is further developed by a treatment of A-pairs—abstractly defined pairs, exemplified by allelomorphic pairs of genes in the same cell.

Genetics, naturally the next topic, is well outlined—some of the concepts are Mendelian classes of zygotes, A-classes of gene-pairs, a-classes of genes, in each case classes of genetically related objects. The probability calculus of heredity is briefly developed. Woodger points out that Mendelian theory cannot take account of the "discontinuous" components of a cell—those, such as the cytoplasm, which are not certainly transmitted to all descendants of a cell.

Of embryology and taxonomy Woodger does not pretend to give more than a sketch. A requisite in a logical study of the former would seem to be a clear distinction between embryo and adult. The author sets the division at the instant after which no further structural complexity develops. But, as structural complexity is one of the undefined concepts, and is not even furnished with adequate axioms, we gain little. There is, however, a careful description of various phases of the development of an organism. In taxonomy Woodger distinguishes between varieties, species, and larger groups, substantially on the basis of sterility and epochs of differentiation, those distinctive features which fit most readily into his present theory.

Of course this book is but a beginning of the axiomatic treatment of biology. For one thing, it holds to topology, avoiding metric statements of size, duration, spatial separation. Again, it restricts definitions to typical cases, which are by no means universal. Cell division is always taken to be a one-two relation. "A woman who gives birth to identical twins does not stand in the relation of sexual parenthood to either twin in the sense here defined." Likewise, continuous components of zygdendra are forbidden to divide in precisely the second "generation" before a gamete. As this, though typical, is not universal, a mathematician might well prefer to say "precisely the nth generation."

There have been mistakes in biological reasoning in the past. The reviewer would be glad to have cases cited in which the existence of Mr. Woodger's calculus would have prevented errors.* Once mastered by biologists, it may well help them to more rapid, reliable reasoning in the future.

E. S. ALLEN

Introduction to the Theory of Fourier Integrals. By E. C. Titchmarsh. Oxford, Clarendon Press, 1937. 10+390 pp.

The theory of Fourier integrals, although originating as early as that of Fourier series, has not been adequately treated in monograph form until recently, when "introductions" to the subject were published by Bochner, Wiener, and Zygmund (a chapter in his excellent book on trigonometric series). The method of Fourier trans-

* Cf. R. A. Fisher, loc. cit., p. 7: "It is a remarkable fact that had any thinker in the middle of the nineteenth century undertaken, as a piece of abstract and theoretical analysis, the task of constructing a particulate theory of inheritance, he would have been led, on the basis of a few very simple assumptions, to produce a system identical with the modern scheme of Mendelian or factorial inheritance."

forms and of more general Fourier-Stieltjes transforms (and also of related Laplace, Mellin, Hankel, Watson, · · · , transforms) on the one hand is becoming an almost universal tool for treating various problems arising in the theory of functions of a complex variable, theory of linear operators, harmonic analysis, probabilities, mathematical physics; on the other hand it offers an inexhaustible source of formulas and relations which are interesting, or at least curious, in themselves, irrespective of possible applications. The present monograph, which hardly can be considered as an "introduction," centers its attention mainly in the latter aspect of the theory of Fourier integrals and presents a wealth of interesting material, a considerable portion of which is due to most recent investigations. A partial list of contents follows.

Chapter I (Convergence and summability, pp. 1–49) treats of formal aspects of Fourier, Laplace, and Mellin transforms, and gives fundamental results concerning convergence and summability (Cesáro, Cauchy, Poisson, Weierstrass) of the corresponding integrals. Chapter II (Auxiliary formulae, pp. 50–68) gives a preliminary survey of the Parseval formula, theory of convolution (Faltung) of Fourier transforms, and Poisson summation formula. Chapter III (Transforms of the class L^2, pp. 69–95) is devoted to a treatment of Fourier transforms in L^2 together with some related topics. Results of this chapter are partially extended to transforms in L^p in Chapter IV (Transforms of other L-classes, pp. 96–118). The theory of conjugate trigonometric integrals and the closely related theory of Hilbert transforms is dealt with in Chapter V (Conjugate integrals, Hilbert transforms, pp. 119–151), while the next, Chapter VI (Uniqueness and miscellaneous theorems, pp. 152–176), in addition to the problem of unique representation, treats of some refinements of the Parseval formula and problems of growth of Fourier transforms. Chapter VII (Examples and applications, pp. 177–211) contains a considerable number of special formulas involving Fourier integrals. The theory of "general transforms," which was originated recently by Watson, and the theory of "self-reciprocal" functions with their various generalizations are discussed in Chapter VIII (General transforms, pp. 212–244) and Chapter IX (Self-reciprocal functions, pp. 245–274). Next, Chapter X (Differential and difference equations, pp. 275–302) contains numerous special applications to the theory of difference and differential equations, giving a rigorous exposition of some parts of the theory known under the name of "operational calculus." The last chapter, XI (Integral equations, pp. 303–369), deals with a considerable number of special integral equations. The book closes with a substantial bibliography (pp. 370–387) and a short index.

J. D. TAMARKIN

Methoden und Probleme der dynamischen Meteorologie. By H. Ertel. (Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 5, no. 3.) Berlin, Springer, 1938. 4+122 pp.

The mathematician will perhaps be surprised to learn that the difficulties in the study of the dynamics of our atmosphere are essentially of a mathematical nature. In fact this subject, as well as stellar hydrodynamics, offers a virgin field for the applied mathematician, and it is to be hoped that Ertel's monograph will serve to attract the mathematical skill which meteorology needs.

As can be inferred from the title, Ertel's monograph is not intended to serve as a textbook in meteorology. One important omission is atmospheric turbulence. The problems that are treated have mostly been the subject of Ertel's own researches and here, as in the original papers, the elegance of the treatment may seem a bit luxurious to the practical meteorologist.

A feature which is not found in textbooks is the formulation of the variational

principle in the dynamics of the atmosphere. It is also gratifying to find at last in meteorological literature the name of Kelvin mentioned (p. 58) in connection with the problem of convective instability of moist air, which he treated for the first time in 1865. In his treatment of the heat balance of the atmosphere the author fails to stress a fundamental difficulty, which is our ignorance, at present, of the radiative behaviour of water vapor under atmospheric conditions. The result of Hergessel referred to on page 76, that in a semi-grey atmosphere at radiative equilibrium there would exist a uniform temperature of −54°C, has been shown to be incorrect.*

On the whole the book offers a convenient exposition of present-day problems of dynamic meteorology.

C. L. PEKERIS

Technique de la Méthode des Moindres Carrés. By Henri Mineur. (Monographies des Probabilités, publiées sous la direction de M. Émile Borel, no. 2.) Paris, Gauthier-Villars, 1938. 8+93 pp.

Mineur states in the preface of his book that it should be possible for the reader to learn to use the method of least squares without understanding the theory behind it. In fact, the theory of the various operations is not fully explained until the fifth chapter. For this reason the mathematician will perhaps find it more satisfactory to proceed directly from the first chapter to the fifth and then read the second, third, and fourth. We shall follow this order in discussing the topics which Mineur treats.

The first chapter consists of an example for which it is desired to fit a linear equation to a set of data. The attempt to make such an equation conform exactly leads in general to an incompatible system. The equation which is the best fit in the sense of least squares is shown in Chapter 5 to result from the so-called normal equations which are compatible and linear. The resolution of a linear system by means of determinants is cumbersome and hence the author presents the Gaussian method which constitutes the principal part of the technique. The method of least squares is shown to be equivalent to finding the equation which produces the minimum probable error. Also in Chapter 5 one finds a discussion of such concepts as mean, standard deviation, probable error of a single measurement, and probable error of the mean of a set of measurements. In fact the author gives a brief but clear exposition of the elements of statistics. It is, however, surprising that simple, multiple, and partial correlation are omitted.

Chapter 2 contains a detailed description of the method of tabulating the data, forming the normal equations, and solving them. Mineur fails to note that much of this tabulation is unnecessary when a computing machine is used. For example, by means of a machine it is possible to obtain a sum of products as a single operation without recording the individual products. In Chapter 3 the author discusses the nature of errors of measurement. He also indicates how the method of least squares can be applied to nonlinear equations. In Chapter 4 he applies his method to the solution of a problem in stellar statistics.

On the whole, this is a readable and useful book.

A. H. COPELAND

British Association for the Advancement of Science: Mathematical Tables. Volume 6: *Bessel Functions.* Part 1: *Functions of Orders Zero and Unity.* Cambridge, University Press; New York, Macmillan, 1937. 20+288 pp.

The preface to this volume opens with the words: "It is with the satisfaction of

* C. L. Pekeris, Gerlands Beiträge, vol. 28 (1930), p. 377.

keeping a long-anticipated engagement that a Committee of the British Association issues its first volume of tables of Bessel functions. Half a century ago, the Committee decided that the tabulation of Bessel functions was the most useful undertaking that it could promote." In fact, one of the tables in the *Treatise of Bessel Functions*, by Gray and Matthews, is credited to the Reports of the British Association for 1889, and in the revised edition of 1922 the Committee is quoted as intending "to publish at an early date a volume of fairly complete tables of Bessel functions."

But with the years the undertaking has grown until we now have one volume published, a second, on functions of other integral orders, in "an advanced state of preparation," and a third taking shape.

One hundred and seventy pages are devoted to the functions $J_0(x)$ and $J_1(x)$, which are tabulated to ten places of decimals (with second differences) at intervals of 0.001 for x from $x=0$ to $x=16$ and at intervals of 0.01 for x from $x=16$ to $x=25$. Then follow tables for the zeros of $J_0(x)$ and $J_1(x)$, the values of the functions $Y_0(x)$, $Y_1(x)$, $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$, together with auxiliary functions and coefficients useful in interpolation. The definitions of these functions through differential equations, power series, and recurrence formulas are conveniently and concisely given.

Mechanically the tables should be easy to use. The entries are in blocks of five with the integral part of each entry given only at the head of the block, unless a change in "characteristic" requires more frequent entry. In the opinion of the reviewer, this is an ideal arrangement for tables of this sort.

The comparisons made with other tables and the listing of errors found in earlier tables are both interesting and valuable.

The Committee has had the courage to say, "There is thus every reason to believe that the tables are completely free from error." Let us hope that nothing more serious has crept in than the trivial missing decimal point in the value of $J_1(7.840)$ which the eye casually picks up on page 80.

The British Association is making a valuable contribution in preparing these authoritative source books of numerical tables. The future volumes in the series on Bessel functions will be awaited with interest.

G. R. CLEMENTS

Essai sur les Fondements de la Géométrie Euclidienne. By Julien Malengreau. Lausanne, Payot, 1938. 311 pp.

This book is intended as an introduction to a more complete treatise on geometry. The author gives a set of twenty-nine postulates, involving the undefined terms *point* and *distance*, and develops the resulting theory.

The problems of the consistency and independence of the postulates are not considered. As a matter of fact it is fairly obvious that rational euclidean 3-dimensional geometry R_3 (that is, the subset of a cartesian 3-space consisting of those points all of whose coordinates are rational) is an example of a space in which postulates 1 to 28 are satisfied. Postulate 29 (il existe au moins un pentapoint parfait) requires that the space be at least 4-dimensional, and 1–29 hold true in R_4.

There is no axiom of *continuity* (or *completeness*) such as the Dedekind cut axiom. On the other hand there is no closure axiom limiting the set of points on a line to a countable set. Likewise, there is no closure axiom limiting the dimension.

The treatment is elementary, as the proofs are all based on rational arithmetic. There are fifty-six figures which illustrate many of the postulates and theorems.

J. H. ROBERTS

NOTES

In addition to the members of the editorial staff of the Bulletin, the following persons have assisted the editors, either by serving as referees or by advising concerning papers offered for publication in the present volume: R. P. Agnew, A. A. Albert, R. G. Archibald, Reinhold Baer, M. A. Basoco, Harry Bateman, E. F. Beckenbach, A. C. Berry, H. R. Brahana, Richard Brauer, E. W. Chittenden, J. A. Clarkson, A. B. Coble, Abraham Cohen, Richard Courant, H. B. Curry, J. L. Doob, Nelson Dunford, W. L. Duren, Arnold Emch, G. C. Evans, L. R. Ford, W. B. Ford, Philip Franklin, T. C. Fry, H. L. Garabedian, H. H. Goldstine, Saul Gorn, W. C. Graustein, J. A. Greenwood, V. G. Grove, W. A. Hurwitz, Dunham Jackson, Nathan Jacobson, R. D. James, R. L. Jeffery, E. P. Lane, R. E. Langer, Lincoln La Paz, D. H. Lehmer, D. N. Lehmer, Walter Leighton, Norman Levinson, Hans Lewy, Saunders MacLane, H. H. Mitchell, Deane Montgomery, C. N. Moore, Marston Morse, Oystein Ore, A. E. Pitcher, H. A. Rademacher, G. Y. Rainich, Wladimir Seidel, W. E. Sewell, I. M. Sheffer, J. A. Shohat, Virgil Snyder, D. J. Struik, Otto Szász, Gabor Szegö, J. M. Thomas, T. Y. Thomas, W. J. Trjitzinsky, H. S. Vandiver, R. J. Walker, J. L. Walsh, S. E. Warschanski, Hermann Weyl, G. T. Whyburn, D. V. Widder, Norbert Wiener, John Williamson, Leo Zippin. The editors desire publicity to acknowledge this service.

The Fifth International Congress for the Unity of Science is to be held at Harvard University, September 5–10, 1939. The theme of the Congress is *Logic of Science*; interest will center upon the relation of concepts, laws, and methods of the various sciences. Attention will be devoted to general problems connected with the unification of science, and, in particular, with the logic of the physical sciences, the relation of the physical and biological sciences, and the relation of the biological and socio-humanistic sciences. There will also be a number of special sessions and symposia concerned with special problems and fields. The Congress is sponsored by the International Committee of the Congresses for the Unity of Science, by the International Institute for the Unity of Science, and, in America, by the American Association for the Advancement of Science, the Philosophy of Science Association, the Association for Symbolic Logic, and the American Philosophical Association. A series of twenty monographs, entitled *Foundations of the Unity of Science* (and constituting the first two volumes of the *International Encyclopedia of Unified Science*) is now being issued by the University of Chicago Press, and helps to provide a background for the Congress. Three monographs have already appeared, and it is hoped that all twenty will be in print by the opening of the Congress. Those who wish later notices of the Congress are requested to send their name and address to Professor C. W. Morris, University of Chicago, Chicago, Illinois.

On the occasion of the Society's Semicentennial Celebration two volumes were published. Volume I, written by R. C. Archibald, is entitled *A Semicentennial History of the American Mathematical Society 1888–1938, with Biographies and Bibliographies of the Past Presidents*. The price is $3.00. Volume II, *Semicentennial Addresses*, contains papers by E. T. Bell, J. F. Ritt, Norbert Wiener, E. J. McShane, T. Y. Thomas, R. L. Wilder, G. C. Evans, J. L. Synge, and G. D. Birkhoff. The price of this volume of more than three hundred pages is $4.50. Members of the Society may obtain both

these volumes until December 1 at a special price of $5.00. Subsequently the usual 25 per cent discount will be allowed to members.

In view of the great losses of books and literary material sustained by Chinese institutions of higher learning because of the Japanese invasion, the American Library Association has issued an appeal to libraries and to individuals for contributions of books and periodicals. Scientific publications will be especially welcome. Further information regarding this project, and instructions for the shipment of material, can be obtained by writing to the Secretary of the American Library Association, 520 North Michigan Avenue, Chicago, Ill.

N. V. Swets and Zeitlinger, Keizersgracht 471, Amsterdam (C), Holland, announce that they have purchased the stock of back volumes 1–80 of the Mathematische Annalen and are ready to reprint those numbers now out of print. The announced price is $480, including index and library binding. Reprinting is contingent on a sufficient number of orders being received. Institutions desiring information should correspond with Swets and Zeitlinger. Volumes 81 and following are in print and can be supplied. One bound set of volumes 1–80 is immediately available at $650.

The School of Mathematics of the Institute for Advanced Study each year allocates a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics of the Institute, Fine Hall, Princeton, N. J., and are returnable by February 1, 1939.

Among the Americans to address the meeting this summer of the British Association for the Advancement of Science were Dean G. D. Birkhoff and Professor J. H. Van Vleck, of Harvard University.

Dr. B. Ramamurti and Dr. S. Sivasankaranarayana Pillai, both of Annamalai University, have been awarded the Ramanujan Memorial Prize of Madras University.

The Keith Prize of the Royal Society of Edinburgh has been awarded to Dr. H. S. Ruse, professor of pure and applied mathematics at University College, Southampton, for his paper *On the geometry of Dirac's equations and their expression in tensor form*, and for other papers.

Dr. R. L. Sackett, formerly dean of the school of engineering at Pennsylvania State College, has been awarded the Lamme Medal of the Society for the Promotion of Engineering Education for achievement in this technical field.

Leverhulme fellowships have been awarded to Dr. S. Goldstein, lecturer in mathematics at the University of Cambridge, and Mr. R. Wilson, lecturer in pure and applied mathematics at University College, Swansea.

Dr. F. C. Ferry, president of Hamilton College, has retired with the title emeritus.

Professor Arnaud Denjoy, of the University of Paris, is in residence at Harvard University for the first half of the academic year 1938–1939 as an exchange professor from France.

Professor Marston Morse, of the Institute for Advanced Study, lectured this summer at the Institut Henri Poincaré, before the Société Mathématique de France, and at Hamburg, Frankfort, Muenster, Jena, and Göttingen.

Professor Emeritus Virgil Snyder, of Cornell University, was honored on September 9 at a dinner given by the mathematicians who have written Ph.D. theses under his direction. A silver bowl appropriately engraved was presented to him.

Dr. V. G. Iyer has been appointed lecturer in mathematics in the Hindu College, Masulipatam.

Associate Professor L. W. Cohen, of the University of Kentucky, has been promoted to a professorship. Professor Cohen is on leave of absence during 1938–1939, and will be at Brown University.

Associate Professor J. L. Dorroh, of Judson College, Marion, Alabama, has been appointed to a professorship of mathematics and physics at Ouachita College, Arkadelphia, Arkansas.

Assistant Professor F. C. Gentry, of the University of Oklahoma, has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Associate Professor B. P. Gill, of the College of the City of New York, has been promoted to a professorship.

Dr. J. A. Greenwood has been promoted to an assistant professorship at Duke University.

Assistant Professor Lois W. Griffiths, of Northwestern University, has been promoted to an associate professorship.

Dr. E. A. Hedberg has been appointed to an associate professorship at Baylor University.

Dr. Nathan Jacobson, of the University of North Carolina, has been promoted to an assistant professorship.

Assistant Professor M. S. Knebelman, of Princeton University, has been appointed professor and chairman of the department of mathematics at the State College of Washington.

Dr. J. K. L. MacDonald has been appointed to an assistant professorship at Cooper Union.

Mr. C. E. Marshall, of the University of Illinois, has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

In the September, 1938, List of Officers and Members of the American Mathematical Society, Mr. Kenneth May should be listed as a fellow of the Institute of Current World Affairs, New York City.

Dr. J. F. Randolph has been appointed to an assistant professorship at Cornell University.

Mr. N. N. Royall, Jr., has been appointed to an assistant professorship at The Citadel.

Mr. T. C. Sermon, of the Michigan College of Mining and Technology, has been promoted to an assistant professorship.

Assistant Professor J. H. Service, of the Michigan College of Mining and Technology, has been promoted to an associate professorship.

Associate Professor H. M. Sheffer, of Harvard University, has been promoted to a professorship.

Assistant Professor C. G. Stipe, of the Michigan College of Mining and Technology, has been promoted to an associate professorship.

Assistant Professor W. G. Warnock, of Fort Hays Kansas State College, has been appointed to an assistant professorship at the University of Alabama.

Dr. K. W. Wegner, of the University of Minnesota, has been appointed to a professorship at the College of St. Catherine, St. Paul, Minnesota.

The following members of the American Mathematical Society are in residence at the Institute for Advanced Study for all or part of this academic year: Professor H. E. Arnold, of Wesleyan University; Dr. P. G. Bergmann; Dr. Herbert Busemann; Professor H. B. Curry, of Pennsylvania State College; Professor Jesse Douglas; Dr. Kurt Gödel; Professor G. A. Hedlund, of Bryn Mawr College; Dr. Witold Hurewicz; Professor B. W. Jones, of Cornell University; Dr. Dorothy Manning; Professor Walther Mayer; Dr. A. P. Morse; Professor Tadasi Nakayama of Osaka Imperial University; Dr. Hyman Serbin; Mrs. C. C. Torrance; Dr. C. C. Torrance; Dr. Henry Wallman.

The following persons are also to be at the Institute for Advanced Study during a part or all of the current academic year: Dr. Valentin Bergmann, Dr. W. H. Barkas, Professor Niels Bohr, Professor Claude Chevalley, Dr. Paul Erdös, Professor I. I. Rabi, Dr. H. E. Robbins, Professor Leon Rosenfeld, Miss Gertrude K. Stanley, Mr. J. S. de Wet.

The following appointments to instructorships are announced: University of Alabama: Dr. C. R. Cassity; Brooklyn College (Evening Session): Dr. J. J. DeCicco; College of the City of New York (Evening Session): Dr. Max Shiffman; University of Illinois: Dr. P. R. Halmos; University of Kentucky: Mr. J. M. Boswell, Mr. V. W. Pfeiffer; Keuka College: Mr. H. F. Archibald; Massachusetts Institute of Technology: Dr. A. H. Clifford; University of Michigan: Miss Edith R. Schneckenburger; University of Minnesota: Dr. H. H. Campaigne; University of Nebraska: Dr. D. M. Dribin; University of North Carolina: Mr. R. E. Smith; Pennsylvania State College Undergraduate Center (DuBois): Dr. M. E. Shanks; Purdue University: Dr. R. H. Downing; Queens College: Dr. Philip Hartman; Rutgers University: Dr. P. T. Maker; San Antonio Junior College: Mr. L. H. Tulloch; University of Texas: Mr. W. B. Coleman, Dr. H. S. Kaltenborn; United States Naval Academy: Dr. H. C. Ayres.

Dr. H. Fitting, of Königsberg, died June 15, 1938.

Professor Emeritus D. N. Lehmer, of the University of California, died on September 8, 1938, at the age of seventy-one. He had been a member of the Society since 1900. A more adequate notice will appear in a later issue.

The death of Professor J. J. Luck, of the University of Virginia, has been reported. Professor Luck had been a member of the Society since 1916.

Professor S. A. Stephenson, of Rutgers University, died on September 1, 1938, at the age of sixty years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

406. H. W. Brinkmann: *Relations between zeta-functions of different algebraic fields.*

Let $k_1, k_2, \cdots, k_r$ be algebraic fields and $\zeta_i(s)$ the zeta-function of k_i. This paper deals with relations of the form $\zeta_1^{a_1}\zeta_2^{a_2}\cdots\zeta_r^{a_r}=1$, where a_i are integers (positive or negative). It is shown how, for given fields k_i, all such relations can be found. Several applications are given. (Received September 27, 1938.)

407. Richard Courant: *Conformal mapping of non-orientable surfaces on plane domains.*

Riemannian manifolds with finite characteristic number but possibly with infinitely many boundaries can be mapped conformally on a plane slit domain so that cross-caps or handles are represented by proper coordination of edges of slits in the plane. This result has been applied by the author to the solution of Plateau's problem for arbitrary topological structure. The present paper is an extension to one-sided surfaces of former results obtained for orientable surfaces. The method consists in a topological discussion of stream lines of an analytic function on the surface with one pole. A detailed proof is contained in a paper on conformal mapping to appear in the American Journal of Mathematics. (Received September 24, 1938.)

408. Richard Courant: *New remarks on Plateau's problem.*

The problem of the existence of minimal surfaces of given topological structure under general boundary conditions is solved by new devices on the basis of a variational problem of the Dirichlet type. In the case of genus zero the parameter domain is a Riemann surface consisting of k unit circles connected in $2k-2$ branch points. For higher genus a corresponding number of full planes are attached in 4 branch points each. After the variational problem has been solved, the character of the solution as a minimal surface is an immediate consequence of the variational conditions for the k boundary circles and the branch points. The possibility of conformal mapping of a k-fold connected domain on a k-fold unit circle with $2k-2$ branch points is a side result. For higher topological structure, the intrinsic advantage of using mapping theorems is discussed. (Received September 27, 1938.)

409. F. A. Ficken: *The Riemannian and affine differential geometry of product-spaces.*

The object of this paper is to discuss systematically the Riemannian (or affine)

differential geometry of the topological product of two or more Riemann (or affinely connected) spaces. In the direct (orthogonal) product of two Riemann spaces, product-tensors and product-connections are defined and some of their properties are given. After several general theorems are proved, results are presented on the following topics: geodesic properties of subspaces, parallel displacement and curvature, parallel fields of vector spaces, and motions. Certain of these theorems are then extended to the direct (affine) product of two affinely connected manifolds. Returning to the product of two Riemann spaces, which is no longer assumed to be direct, after proving several general theorems the author shows, finally, that if such a space has any one of a number of simple properties, it is necessarily the direct product of its factors. (Received September 19, 1938.)

410. Tomlinson Fort: *The Euler-Maclaurin summation formula.*

The classical Euler-Maclaurin sum formula expresses a sum as an integral plus terms involving the Bernoulli numbers. The present paper gives an extension to n-fold sums which are expressed as integrals plus terms involving the Bernoulli numbers of higher order as defined by Nörlund. If $n=1$, the formula reduces to the classical formula and the remainder formula developed in the paper reduces to the classical Jacobi form. (Received September 28, 1938.)

411. D. W. Hall and A. D. Wallace: *Some invariants under monotone transformations.*

Let A be a locally connected continuum and let P be a property of point sets. If P is the property of being connected, it has been shown (Kuratowski, Fundamenta Mathematicae, vols. 8 and 13) that the following properties are equivalent. $\Delta_0(P)$: If A is the sum of two continua, their product has property P. $\Delta_1(P)$: The boundary of a component of the complement of a continuum has property P. $\Delta_2(P)$: Each irreducible separation of A between two points has property P. $\Delta_3(P)$: If the sets X and Y are disjoint closed sets containing the points x and y, then there exists a closed set having property P that separates A between x and y and is disjoint with both X and Y. In this note, the following theorems are proved: (1) for each $i=0, 1, 2$ and any property P of point sets, $\Delta_i(P)$ implies $\Delta_{i+1}(P)$. (2) If the property P is invariant under monotone transformations, so also is $\Delta_i(P)$. (Received September 29, 1938.)

412. T. R. Hollcroft: *Curve systems with distinct nodes and cusps and of negative virtual dimension.*

Plane sections of tangent cones to a family of singular surfaces in S_3 with only a cuspidal curve in common constitute an irregular plane curve system whose only singularities are distinct nodes and cusps. This system is defined by two equations in one parameter. The virtual dimension of this system is negative for high orders of the surface. In S_r the tangent cone to a nonsingular primal V_{r-1} of order ν from an S_{r-2} cuts V_{r-1} in the contour curves Γ. From S_{r-3}, the system Γ projects into a system of plane curves C which is irregular for the following limiting values of r and ν: $r=3$, $\nu\geqq 5$; $r\geqq 4$, $\nu\geqq 3$. This system C is defined by a set of three equations in two parameters. Its only singularities are distinct nodes and cusps. The system C has a negative virtual dimension for the following limiting values of r and ν: $r=4$, $\nu\geqq 8$; $r=5$, $\nu\geqq 5$; $r=6, 7, 8$, $\nu\geqq 4$; $r\geqq 9$, $\nu\geqq 3$. The above systems may in some cases be reducible. (Received October 1, 1938.)

413. A. N. Lowan: *On Green's functions in the theory of heat conduction in spherical coordinates.*

This paper is a sequel to the paper *On the operational determination of two dimensional Green's functions in the theory of heat conduction* (this Bulletin, vol. 44 (1938), pp. 125–133). Expressions for Green's functions for a sphere, and for the infinite solid bounded internally by a sphere, are obtained. The corresponding Laplace transforms are derived in a form which makes it possible to use the results of the previous paper. The inversion of these transforms is then accomplished as before. (Received September 19, 1938.)

414. A. N. Lowan: *On the computation of the second difference of the $Si(x)$, $Ci(x)$, and $Ei(x)$ functions.*

(1) Define $R(x)=[\phi(x+h)+\phi(x-h)-2\phi(x)]-[(h/2)\phi'(x+h)-(h/2)\phi'(x-h)]$; whence (2) $|R(x)|<\sum_{k=2}^{\infty}[h^{2k}/(2k-1)!]\{\phi^{(2k)}(x)\}$ where $\phi(x)$ is any of the above functions and $\{\phi^{(2k)}(x)\}$ an upper bound of the modulus of its $2k$th derivative. The latter is expressed as a definite integral, and its upper bound is found. For x larger than a certain x_0 and $h=10^{-4}$, it is shown that the upper bounds of $R(x)$ for each of the functions under consideration is of the order of magnitude of 5 times 10^{-12}. The first bracket in (1), which represents the second difference of $\phi(x)$, may therefore be approximated by the second term in (1), with an error not exceeding 5 times 10^{-12}. This result is applied as a check on the computation of $Si(x)$, $Ci(x)$, and $Ei(x)$, for the range between 0 and 2 at intervals of 10^{-4}, in progress now by the W.P.A. Project for the Computation of Mathematical Tables, sponsored by Dr. Lyman J. Briggs, Director of the National Bureau of Standards, Washington, D. C. (Received September 19, 1938.)

415. J. D. Mancill: *Problems of the calculus of variations with prescribed transversality conditions.*

Problems of the calculus of variations in $(x, y_1, y_2, \cdots, y_n)$ space for which a prescribed relation exists between the directions of the extremals and the transversal directions were studied first by Rawles (Transactions of this Society, vol. 30 (1928), pp. 765–784). More recently, LaPaz, using a method and point of view quite different from that of Rawles, has given a rather complete treatment of the problem in non-parametric form (this Bulletin, vol. 36 (1930), pp. 674–680). In the present paper, a method similar to that of LaPaz is used, which avoids, however, his very intricate treatment of an associated system of nonhomogeneous partial differential equations by reducing the problem to a very simple total differential equation. The method applies with equal facility to parametric and non-parametric problems of the calculus of variations in space of any number of dimensions. The problem is treated in parametric form. Necessary and sufficient conditions in order that a transversality relation belong to a problem of the calculus of variations are derived. Finally, there is obtained the most general integrand function of a problem of the calculus of variations to which a given transversality relation belongs. (Received September 2, 1938.)

416. J. A. Shohat: *On the generalized orthogonal polynomials.*

Consider a complete (that is, of all possible degrees $n=0, 1, 2, \cdots$) sequence of polynomials $\{\Phi_n(x)\}$ with highest coefficient unity. It is shown, on the basis of a recently established theorem of R. P. Boas, that a necessary and sufficient condition for the said sequence to form an orthogonal system with respect to infinitely many

weight functions $\psi(x)$ of bounded variation in $(-\infty, \infty)$ is: $\phi_{n+2}(x)=(x-c_{n+2})\Phi_{n+1}(x)-\lambda_{n+2}\Phi_n(x)$, $(n=0, 1, \cdots)$, with all λ_n different from zero. For $\lambda_n>0$, $\{\Phi_n(x)\}$ is an ordinary sequence of orthogonal polynomials, that is, the weight function $\psi(x)$ is monotone and non-decreasing in $(-\infty, \infty)$. (Received September 29, 1938.)

417. Louis Weisner: *A characteristic property of completely decomposable groups.*

A completely decomposable group is one which is the direct product of simple groups. The author proves that a finite composite group is completely decomposable if, and only if, the cross cut of its maximal invariant subgroups is the identity. (Received September 2, 1938.)

418. H. A. Arnold: *Topological order in Kantorovitch spaces.*

Consider a partially ordered space S of Kantorovitch with elements $x, y, z, \cdots$. Given a completely continuous transformation $F(x)$ transforming the closure $\overline{W}$ of an open bounded subset W into a subset of S, there exists a function $F_\epsilon(x)$ and a fixed y in S such that $|F(x)-F_\epsilon(x)|<\epsilon y$ for all x in $\overline{W}$, and such that the values of $F(x)$ are all in a fixed finite-dimensional linear subspace of S. The definitions of topological degree and index follow readily. It is easily proved that compact sets are bounded. Necessary and sufficient conditions are derived, in lattice language, that the space be bicompact; that for every sequence $\{p_i\}$ such that $p_i\to 0$ we have $|p_i|<\epsilon y$ for $i>N$ and y fixed throughout; that to a given compact set there exist a fixed y such that to any ϵ there are elements x_i finite in number with the property that, for every other element x of the set, $|x-x_i|<\epsilon y$ for at least one i. Necessary and sufficient conditions are derived that these properties be pairwise equivalent. (Received October 29, 1938.)

419. Clifford Bell: *Plane curves with pseudo rhamphoid cusps.*

In a previous paper (American Mathematical Monthly, vol. 64 (1937), pp. 218–221) the author developed methods of determining the parameters of the cusps and of those points for which the tangent line has more than two point contact with the curve. The parametric equations used were assumed to give a proper parametric representation of the curve. In this paper those curves are considered whose parametric equations improperly represent other curves. Consider a curve as being traced out by a moving point whose motion is governed by the parameter. A pseudo cusp on such a curve is defined as a point at which the moving point stops, reverses its direction, and then moves back along the path upon which it arrived. Such pseudo cusps, together with the ordinary cusps, are obtained by the methods of the previous paper when applied to the above mentioned curves, whose parametric equations improperly represent other curves. (Received October 26, 1938.)

420. R. P. Dilworth: *Archimedean residuated lattices.*

Let Σ be a noncommutative residuated lattice in which both the ascending and descending chain conditions hold. The union m of all elements x such that $x^\sigma=z$ for some σ is called the *radical* of Σ. If $m=z$, Σ is said to be *semisimple*. Elements which cover the null element z are called *simple*. Let Σ' be the lattice of elements a such that $a\supset m$, and define a multiplication over Σ' in terms of the residuals. Then Σ' is semisimple. Let Σ also be semisimple. Then if each element of Σ can be represented as a union of simple elements, Σ is a Boolean algebra. If Σ is an arbitrary modular, semisimple lattice, it is shown that the simple elements of Σ generate a Boolean algebra Σ_B which is dense in Σ. Other results of a similar nature are obtained, and the structure

of the lattice in the vicinity of the unit element is determined. (Received October 28, 1938.)

421. D. W. Hall: *On pointwise periodic homeomorphisms.*

In this note the essential features of an example by Ralph Phillips and W. L. Ayres (see a forthcoming article in Fundamenta Mathematicae) are combined with those of an example by G. E. Schweigert and the author (Duke Mathematical Journal, December, 1938) to prove the following theorem: Let r_i, $(i=1, 2, 3, \cdots)$, be any sequence of positive integers containing the integer 1. Then there exists in four-dimensional Euclidean space a locally connected continuum M and a pointwise periodic homeomorphism $T(M)=M$ having the following properties: (a) there exists in M an infinite sequence of orbits under T converging to a limit set L such that for every i some free arc of L has no interior point of period different from r_i; (b) the closure of every component of $M-L$ is a 2-cell. (Received October 28, 1938.)

422. H. J. Hamilton: *Preservation of partial limits in multiple sequence transformations.*

Consider the problems, for various pairs of classes X and Y, of determining conditions on the matrix $\|a_{pqij}\|$ necessary and sufficient that the double sequence $\{\sigma_{pq}\}$ be of class Y, where $\sigma_{pq}=\sum a_{pqij}s_{ij}$, whenever the double sequence $\{s_{ij}\}$ is of class X, with $\lim_p \sigma_{pq}=\lim_i s_{ij}$ and $\lim_q \sigma_{pq}=\lim_j s_{ij}$ (1) for all q, p sufficiently large, (2) for all q, p. Analogous problems are solved for sequences of dimension n in general. (Received October 28, 1938.)

423. O. G. Harrold: *On hereditary arc sums.*

A continuous curve M each of whose sub-continua is an arc sum will be called an hereditary arc sum. In this paper it is proved that this property is equivalent to each of the following: (1) each acyclic curve in M is an arc sum, (2) the M boundary of an arbitrary region in M is countable. It is shown that every curve containing a continuum of convergence contains an acyclic non-arc sum. Thus the class of hereditary arc sums is contained in the class of hereditarily locally connected continua. (Received September 3, 1938.)

424. O. G. Harrold: *Concerning* $2-1$ *transformations on an arc.*

Let T be a continuous transformation defined on the compact locally connected continuum A. If $B=T(A)$ is such that each point x in B has exactly two inverse points, B is not an arc. If A is an arc and T is exactly $2-1$, then B is not a circle. Under the same circumstances, B can contain no triod composed of free arcs. Thus if T is an exactly $2-1$ continuous transformation defined on an arc, the image set is not a graph. (Received September 3, 1938.)

425. R. D. James: *Note on integers which are not sums of three squares.*

It is well known that integers of the form $4^m(8n+7)$ are not sums of three squares. In this note it is shown that every sufficiently large integer $4^m(8n+7)$ can be written in the form x^2+N, where exactly one prime p congruent to 3 (mod 4) divides N. All the other prime factors of N are thus sums of two squares. In this sense the integers $4^m(8n+7)$ just fail to be representable as a sum of three squares. (Received October 28, 1938.)

426. A. D. Michal: *General Riemannian differential geometry with abstract coordinates and intrinsic inner product.*

The element of arc length in the author's previous studies on general Riemannian geometry is of the form $ds = [\delta x, g(x, \delta x)]^{1/2}$, where $[x, y]$ is the independently postulated bilinear inner product of the Banach space E of coordinates. The present paper is concerned with general Riemannian spaces whose $ds = [\delta x, g(x, \delta x)]^{1/2}$ is defined in terms of the *intrinsic* "inner" product $[x, g]$ of the Banach space E of coordinates. The function $[x, g]$ is bilinear on $E\overline{E}$ to the real numbers, where $\overline{E}$ is the Banach space of numerically valued linear functions on E. Special attention is given to those general Riemannian spaces that have a constant Riemannian curvature. (Received October 29, 1938.)

427. A. D. Michal and D. H. Hyers: *Normal coordinates in general differential geometrics with inter-space inner product.*

A theory of abstract normal coordinates in a general Michal differential geometry with a linear connection was given by the present authors in Annali della Reale Scuola Normale Superiore, Pisa, (2), vol. 7 (1938), pp. 157–176. In a memoir to appear soon in the Mathematische Annalen, the authors studied in detail the more restrictive case in which there exists an intra-space inner product in the coordinate Banach space. In the present paper the authors show how the theory remains essentially unchanged with an inter-space inner product between two Banach spaces E and E_1. The case $E_1 = \overline{E}$, the space of numerically valued linear functions, is of special interest since then the class $k^{(\infty)}$ of coordinate transformations is identical with the class $K^{(\infty)}$. (Received October 29, 1938.)

428. A. A. Shaw: *Mathematical passages in the works of David the Invincible.*

It is the purpose of this paper to give a faithful translation of all the important mathematical passages in the works of David the philosopher (the distinguished Armenian scholar of the fifth century of our era), together with a critical commentary with special reference to the related topics in Plato and Aristotle. The passages mentioned deal with: (a) theory of numbers of Pythagorean character; (b) definitions and expositions of mathematical concepts; (c) history of mathematics. The paper will also give some account of the life and works of this Armenian philosopher, generally known as David the Invincible, with illustrative photographs from the original manuscript. David occupies the same place in the history of Armenian philosophy as do Plato and Aristotle in the Greek. (Received October 18, 1938.)

429. A. A. Shaw: *An astronomical passage of antiquity.*

This paper deals with and will be a commentary on the following passage from the *Treatise on the Revolutions of the Stars*, by Bardadsan, the celebrated scientist of the second century A.D.: "Two revolutions of Saturn are equivalent to 60 years, five of Jupiter to 60 years, fourteen of Mars to 60 years, sixty of the sun to 60 years, seventy-two of Venus to 60 years, hundred fifty of Mercury to 60 years, seven hundred twenty of the moon to 60 years; and it is the same revolution for all celestial bodies, that is, the time of any one of their revolutions. From which it follows that 100 of these revolutions represent 6000 years as follows: 200 revolutions of Saturn, 500 of Jupiter, 1400 of Mars, 6000 of the sun, 7200 of Venus, 15000 of Mercury, 72000 of the moon."

Bardadsan establishes this calculation to prove that the duration of this world would be only 6000 years. (Received October 18, 1938.)

430. A. E. Taylor: *Reflexive Banach spaces.* I. Preliminary report.

If E is a Banach space we denote by $\mathcal{E}$ the linear topological space (l.t.s.) formed from the elements of E by using the weak neighborhood topology of E relative to its conjugate space E^*. Then $\mathcal{E}^*$ denotes the corresponding space derived from E^*. By $\mathcal{E}_*$ we mean the l.t.s. whose elements are those of E^*, the topology being the weak neighborhood topology of E^* relative to E. Then if E_1 and E_2 are two Banach spaces, and if $\mathcal{E}_1$ is linearly homeomorphic with $\mathcal{E}_{2*}$, E_1 and E_2^* are equivalent, and E_1, E_2 are reflexive (that is, regular in the sense of Hahn). In particular, if E_1 and E_2 are the same space E, and if $\mathcal{E}^*$ is linearly homeomorphic with $\mathcal{E}_*$, then E is reflexive. Likewise E is reflexive if $\mathcal{E}$ is linearly homeomorphic with the l.t.s. $\mathcal{E}_{**}$ (the class E^{**} with weak neighborhood topology relative to E^*). If $\mathcal{E}_1$ is linearly homeomorphic with $\mathcal{E}_2^*$ (instead of with $\mathcal{E}_{2*}$), E_1 and E_2^* are isomorphic, but not necessarily reflexive, for it can happen that the linear mapping of E_2^* on E_1 is not continuous as a function on $\mathcal{E}_{2*}$ to $\mathcal{E}_1$. (Received October 26, 1938.)

431. A. E. Taylor: *Reflexive Banach spaces.* II. Preliminary report.

Let E be a Banach space, E^* its conjugate space. Let $A=(\alpha)$ be a partially ordered set with the Moore-Smith property, that is, A is an index set such as is used in defining Moore-Smith convergence. The space E is said to be closed relative to A if whenever $\{x_\alpha\}$ is a directed set (a function on A to E) whose elements form a bounded set in E, there exists an element y in E for which $\underline{\lim}_\alpha f(x_\alpha) \leqq f(y) \leqq \overline{\lim}_\alpha f(x_\alpha)$ for each f in E^*. Then in order that E be reflexive it is necessary and sufficient that it be closed relative to every A. (E is reflexive if every linear functional on E^* is representable in the form $f(x)$, for some x in E.) The space E is said to be boundedly weakly A-compact if given a bounded directed set $\{x_\alpha\}$ there exists a y in E such that every neighborhood of y (in the weak neighborhood topology of E) contains a cofinal subset of $\{x_\alpha\}$. Then if E is boundedly weakly A-compact, it is closed relative to A. Hence if this is true for every A, then E is reflexive. (Received October 26, 1938.)

432. A. E. Taylor: *Banach spaces whose unit sphere is weakly compact.*

Let E be a Banach space whose unit sphere is weakly compact. Then the following things are true: (1) A necessary and sufficient condition that a series $f_1+f_2+\cdots$ of elements of the conjugate space E^* be unconditionally convergent is that the series $|f_1(x)|+|f_2(x)|+\cdots$ be convergent for each x in E. (2) A linear operation on E to l is completely continuous, and is defined by $\{f_i(x)\}$, where the f_i form an unconditionally convergent series in E^*. (3) If f_n, g are in E^*, $(n=1, 2, \cdots)$, and $f_n(x)\to g(x)$ for each x in E, then g is in the closed linear manifold in E^* determined by $\{f_n\}$. (Received October 26, 1938.)

433. F. A. Valentine: *The problem of Lagrange with finite inequalities as side conditions.*

The problem treated here is a generalization of the problems of unilateral variations in the calculus of variations. The side conditions to be satisfied are taken in the form $\phi_\beta(x, y_1, \cdots, y_n)\geqq 0$, $(\beta=1, \cdots, m<n)$. Necessary conditions analogous to those of Euler, Weierstrass, Clebsch, and Mayer are obtained for a minimizing arc E_{12}. The paper centers around an imbedding theorem for a composite finite minimiz-

ing arc. Such an arc is one which consists of two subarcs E_{12} and E_{22} such that all the functions $\phi_3(x, y)$ are positive on E_{12} except possibly at 3, whereas they are constant along E_{22}. This imbedding theorem enables one to establish a sufficiency theorem for the arc $E_{12}=E_{12}+E_{22}$ by strengthening the above necessary conditions in the usual way. The extension of the above results to arcs which are more general than a composite finite one is then immediate. (Received October 19, 1938.)

434. Max Wyman: *The simultaneous theory of a linear connection, and a non-holonomic linear connection in a general geometry with Banach coordinates.*

A tensor calculus for a Hausdorf space, with coordinates in a Banach space E, and a linear connection has been worked out. (See Michal, *General tensor analysis*, this Bulletin, vol. 43 (1937), pp. 394–401.) In this paper we consider, in addition to E, another Banach space E_1 in which non-holonomic vector fields transform according to the law $\overline{V}=M(x, V(x))$. Therefore $M(x, y)$ is a solvable linear function of y on EE_1 to E_1. Non-holonomic linear connections are defined, and covariant differentials of multilinear forms are treated in the usual manner. A normal coordinate theory for E_1 is established, and we prove that the transformation between non-holonomic vector fields in normal coordinates is linear. Non-holonomic normal vector forms are defined, and several relations of symmetry that they satisfy are given. Some of these results are generalizations of known results for the "n" dimensional space. (See Michal and Botsford, Annali Di Matematica, (4), 1933–1934.) (Received October 29, 1938.)

435. Stefan Bergmann: *On the theory of harmonic functions of three variables.*

Starting from results of a previous paper (Mathematische Annalen, vol. 99 (1928), pp. 629–659, especially §3) the author studies integrals $\int[Fdx+F^*dy+F^{**}dz]$, where $F\equiv F(x, y, z)$, F^*, F^{**} are harmonic functions and $[Fdx+F^*dy+F^{**}dz]$ is a total differential. It is shown that in certain cases there exist relations between such integrals and singularities of F. Also, for every F there exist three harmonic functions $G\equiv G(x, y, z, X, Y, Z)$, G^*, G^{**}, depending on parameters X, Y, Z, such that $F(X, Y, Z)=\int[Gdx+G^*dy+G^{**}dz]$. Finally, certain properties of the representation of the space of three variables x, y, z by functions F, F^*, F^{**} are obtained. (Received October 11, 1938.)

436. Reinhold Baer: *Almost Hamiltonian groups.*

The elements in a group which transform every subgroup into itself form the norm of the group. The theory of the groups with cyclic norm quotient group has been developed in a previous paper. This theory establishes pretty well the relation between the norm of a given group and any element of this group with two essential exceptions: this theory does not give any information if the norm and the element in question generate together either an abelian group or a Hamiltonian group. It is the object of this note to deal with the second of these alternatives under the additional hypothesis that the norm quotient group is abelian. These apparently rather weak assumptions turn out to be very restrictive, and this makes it possible to give a fairly complete theory of this class of groups. (Received October 17, 1938.)

437. Reinhold Baer: *Groups with abelian norm quotient group.*

The norm of a group G consists of all those elements in G which transform every

subgroup of G into itself. It is the object of this note to describe the group of automorphisms which are induced in the norm of G by the elements of G, provided the norm quotient group is abelian. (Received October 17, 1938.)

438. Reinhold Baer: *Nets and groups.*

The combinatorial properties underlying the configuration of three pencils of parallel straight lines in the plane have found their condensation in the concept of *net*. It is the object of this paper to establish the equivalence of the theory of nets with a certain chapter in the theory of groups. (Received October 17, 1938.)

439. M. A. Basoco: *On certain arithmetical functions due to G. Humbert.*

G. Humbert has pointed out (Comptes Rendus, vol. 158 (1914), pp. 220, 294) the existence of a class of entire functions having interesting arithmetical properties. In this paper, Humbert's note is extended and supplemented; a set of functional equations is obtained which serve to define Humbert's functions uniquely. Advantage is taken of the fact that these functions are special cases of certain others which the present author obtained in a former paper (American Journal of Mathematics, vol. 54 (1932), pp. 242–252) to relate them to the theta-functions by means of certain identities. The paraphrases of these are related to the representation of a number as the sum of five squares. Some applications involving the greatest integer function are also given. (Received October 18, 1938.)

440. O. E. Brown: *A least squares machine.* Preliminary report.

This paper is a description of a proposed device for the solution, by mechanical means, of a system of linear algebraic equations. It gives approximate solutions, which, by an easy process of iteration, approach the exact values. While the machine may be employed for any system whatever, within its capacity, it is especially valuable when the system is inconsistent, as is usually the case when the constants of the system are obtained by observation. In this case identical springs, one for each equation, introduce modifications in the constant terms and allow the modified system to be solved. The extent and direction of the distortions of the springs determine the residuals. If Hooke's law is assumed to hold for each spring, it follows that the solution of the modified system satisfies the corresponding system of normal equations and, therefore, the sum of the squares of the residuals is a minimum. (Received October 28, 1938.)

441. R. S. Burington: *On the congruence of matrices and associated circavariant matrices.*

Let A, B, C, and P be matrices with elements over a field. Let $C_{r_1\cdots r_t}^{s_1\cdots s_t}$ denote the matrix obtained from C by deleting rows $r_1\cdots r_t$ and columns $s_1\cdots s_t$. If (1) $B=P'AP$, where P is nonsingular, and if (2) $B_{r_1\cdots r_t}^{s_1\cdots s_t}=P_{r_1\cdots r_t}^{\prime r_1\cdots r_t}A_{r_1\cdots r_t}^{s_1\cdots s_t}P_{r_1\cdots r_t}^{s_1\cdots s_t}$, then $A_{r_1\cdots r_t}^{s_1\cdots s_t}$ is called a *circavariant matrix* of A under the congruence (1). In this paper a theory of circavariant matrices is initiated. General theorems relating to the restrictions that must be imposed on P in order that one or more of the set $A^1, A^2, \cdots, A_{r_1\cdots r_t}^{s_1\cdots s_t}, \cdots$ be circavariant are given. Theorems on the congruence of matrices with P in a modified m-affine space are obtained, together with a set of canonical forms. The relation to the author's previous paper (Transactions of this Society, vol. 38 (1935), pp. 163–176) is discussed. The results of this paper are fundamental to the

construction of a general theory of relative-equivalent electrical networks, to be considered later. (Received October 25, 1938.)

442. Max Coral: *On fields for double integral variation problems.*

The form of the most general field for a multiple integral variation problem has been determined recently by Bliss, Smiley, and Alaoglu. In the case of a double integral, the invariant integral of the field involves a quadratic form in the derivatives of the functions along which the integral is computed. The complete determination of the field demands the determination of the coefficients of this quadratic form, concerning which it is known only that they satisfy a certain system of partial differential equations. In the present paper, the author completes the theory as follows: if the extremals of the field are known, the coefficients of the quadratic form can be determined from them without integration of the partial differential equations, and they are unique up to additive functions which remain constant on each extremal. Also, if a region F is simply covered by a family of extremals, conditions on the family are found which are necessary and sufficient in order that F form a field. (Received October 26, 1938.)

443. H. V. Craig: *On extensors and a euclidean basis for higher order spaces.*

This paper is a continuation of the paper, *On tensors relative to the extended point transformation*, American Journal of Mathematics, vol. 59 (1937). The writer considers arbitrary curves $x^a=x^a(t)$ in an x-space of N dimensions and sets up a correspondence with a euclidean space of L dimensions with rectangular cartesian coordinates y^i. The correspondence is obtained by equations expressing the y's as functions of the x's and their derivatives with respect to t up to order M. If M is zero, a Riemannian subspace is obtained. The geometry of the case $M>0$ is studied, and it is found that extensors play a role somewhat similar to that of tensors in Riemannian geometry. A metric extensor is obtained and from it "Christoffel symbols." Some of the Christoffel symbols of reduced range are tensors, others extensors. Excovariant derivatives (analogous to covariant derivatives) are formed and investigated. Generalizations of certain identities of Finsler geometry are found. (Received October 14, 1938.)

444. J. J. DeCicco: *The analogue of the nine-point circle in the associated Kasner plane.*

This is a continuation of two papers by Kasner (Science, vol. 85 (1937), pp. 480–482, and Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 337–341) and abstract 42-11-398 by Kasner. A *simple horn-set* is the totality of all curves (third order elements) which possess a common point and direction. Let $x=\gamma$, $y=d\gamma/ds$, where γ is the curvature and s is the arc length. A simple horn-set is then called the *associated Kasner plane* K_2, where any point of K_2 is a curve (x, y) of the simple horn-set. The group of conformal transformations induces in K_2 the special affine three-parameter group G_3: $X=mx+h$, $Y=m^2y+k$. In this paper, the triangle geometry of the group G_3 is studied. Many theorems of this geometry are completely analogous to those of euclidean geometry, but the majority are entirely different. The analogs of the circumscribed, inscribed, and nine-point circles are obtained and entirely new relationships between them are discovered. The analogs of the Simpson line, Menelaus' and Ceva's theorems, and the Brocard points are also obtained. (Received October 24, 1938.)

445. J. M. Dobbie: *A note on a convergence proof for Fourier series.*

Stone (Transactions of this Society, vol. 28 (1926), pp. 695–761) proved that the expansion problem associated with the differential system $u^{(n)}+\lambda u=0$, $u^{(r)}(0)-u^{(r)}(1)=0$, $(r=0, 1, 2, \cdots, n-1)$, in which the boundary conditions are regular, gives rise to Fourier series. In this note it is shown how the convergence proof for such a system can be made to depend upon the contour integration proof for the first order problem given by Birkhoff (Transactions of this Society, vol. 9 (1908), pp. 373–395), an extension of which was made to the second order problem by Carman (this Bulletin, vol. 30 (1924), pp. 410–416). (Received October 26, 1938.)

446. Wallace Givens: *Signatures of Lorentz matrices.*

It is first proved that every Lorentz matrix of order n can be expressed as the product of involutory Lorentz matrices having the characteristic equation $(x+1)(x-1)^{n-1}=0$. Such an involution I is called spatial or temporal according as $Q(y)>0$ or <0, where y is a solution of the equations $Iy=-y$ and $Q(x)$ is the invariant quadratic form. Two products of such involutions cannot be equal unless the number of spatial (temporal) involutions in one product is congruent (mod 2) to the number of spatial (temporal) involutions in the other. This gives an algebraic treatment of the signatures of Lorentz matrices which seems more direct than that employing the spin representation. (Received October 28, 1938.)

447. D. K. Kazarinoff: *Remark concerning Desargues' triangle theorem.*

This paper gives a projective generalization of a theorem due to Möbius: "If the plane of one of two triangles in perspective is fixed while the plane of the other rotates about the axis of perspective, then the center of perspective describes a circle in a plane perpendicular to the axis of perspective and the center of the circle lies in the fixed plane." Let $A_1B_1C_1$ be a fixed triangle and A, B, C be any three collinear fixed points chosen, respectively, on B_1C_1, C_1A_1, A_1B_1. Consider any two figures (A_2) and (B_2) homological to each other with the center of homology C and the arbitrary chosen plane of homology π, and a third figure (C_2) homological to (A_2) and (B_2) with the centers B and A, respectively, and the plane π. Our theorem is: If a variable triangle $A_2B_2C_2$ deforms in such a way as to have its vertices as the corresponding points of (A_2), (B_2), (C_2), then $A_2B_2C_2$ remains always in perspective to $A_1B_1C_1$ and the center of perspective D describes a figure (D) which is homological to (A_2), (B_2), (C_2) with the centers of homology A_1, B_1, C_1, respectively, and the plane of homology π. (Received October 28, 1938.)

448. J. F. Kenney: *The regression systems of two sums having random elements in common.*

Let $f(x)$ be the probability function of x, and suppose the mean $\int_{-\infty}^{\infty} xf(x)dx$ exists. A sample of n independent variates is taken from the population represented by $f(x)$ and the sum $y=\sum_1^n x_i$ is formed. From this sample $k<n$ values are chosen at random, and a sample of $m-k$, $(m \leqq n)$, independent variates x_i' is taken from the population $f(x)$. The sum $z=\sum_1^k x_i+\sum_{k+1}^m x_i'$ is then formed. The problem of determining the regression systems of y on z and z on y in the population resulting from repeated samples is discussed elsewhere in the literature by Fischer and others. In the present paper a brief and elegant solution is given by means of characteristic functions. (Received October 28, 1938.)

449. Cornelius Lanczos: *A simple interpolation method for the representation of rugged curves.*

If a function is given in equidistant ordinates, the customary method of interpolation makes use of a difference table. This method fails if an empirical function is so unsmooth ("rugged") that the successive differences show no regularity and do not converge toward zero. Such a function may still be developable into a sufficiently convergent Fourier series, and a simple interpolation formula then holds. The validity of the method presumes that the Fourier series is practically limited to a number of terms not exceeding the number of points in which the function is given. In order to satisfy this condition only functions are considered which vanish at the two end points of the given range. The range is chosen as the half period and only sine terms are used, which permits continuity of function and first derivative and thus provides sufficient convergence. If the given values do not satisfy those boundary conditions, we define $\phi(x)=f(x)-(\alpha+\beta x)$, α and β suitably chosen, and interpolate $\phi(x)$. This gives satisfactory results for empirical curves which are not smooth enough for interpolation by differences and yet not too rugged to prohibit any reasonable interpolation. (Received October 24, 1938.)

450. Saunders MacLane: *Steinitz field towers for modular fields.*

If a field K of characteristic p has a transcendence basis T over a perfect subfield $P \subset K$, then T may be called a separating transcendence basis if every element of K is separable and algebraic over $P(T)$. In an investigation of the structure of complete fields with valuations, Hasse and Schmidt have used, but not proved, a theorem to the effect that, in any imperfect field K, one can select a suitable subfield L such that K can be approximated by a tower of fields S_n where S_n has a separating transcendence basis over L. Furthermore, they assert conditions which imply that $S_n = L(S_{n+1}^p)$. The present paper offers examples disproving the existence of such approximating towers in general, but provides a construction for modified such towers in a large class of cases. (Received October 18, 1938.)

451. J. R. Musselman: *The equation of motion of equal maps.*

The theory of rotation enables us to move an object from one position on a plane to another position on the same plane. The purpose of this paper is to discuss a type of motion which will send an object into three or more positions consecutively on the same plane. Or, if we prefer, we may think of three or more equal objects (for convenience called maps) lying on a plane and ask for the equation of motion which will pick up all of them. The relationships existing among corresponding points and among corresponding lines in the maps are pointed out. (Received October 30, 1938.)

452. C. J. Nesbitt: *Note on primary algebras.*

Let A denote an algebra of finite rank over a field K, and let N be its radical; A is called primary if it possesses a unit element and the quotient algebra A/N is simple. Methods in the theory of regular representations of algebras (see Brauer and Nesbitt, Proceedings of the National Academy of Sciences, vol. 23) are here applied to a primary algebra A over an algebraically closed field K to obtain a reduced form for the regular representation of A. By means of this reduced form primary algebras which are also symmetric (see the above reference and Nakayama and Nesbitt, Annals of Mathematics, (2), vol. 39 (1938), p. 659) are studied and, in particular, the linear functions Φ of the elements of A with the property $\Phi(\alpha\beta)=\Phi(\beta\alpha)$, for any

α, β belonging to A, are discussed. The results have bearing on the modular representations of finite groups. (Received October 28, 1938.)

453. Rufus Oldenburger: *Higher-dimensional determinants.*

It has not been known that higher dimensional determinants arise in simple ways in the theory of ordinary determinants. In this paper we show that the coefficients in the expansion of an ordinary determinant whose elements are linear forms are three-dimensional determinants. Also, the derivative of a determinant whose elements are differentiable functions of x with respect to x is a three-dimensional determinant. In analogous ways $(p+1)$-dimensional determinants arise in the study of p-dimensional determinants. From these considerations it follows that invariant ranks of forms defined elsewhere by the author in terms of the vanishing and nonvanishing of p-way determinants may be defined in terms of maximum ranks of $(p-1)$-way generalized Hessian matrices. (Received October 26, 1938.)

454. A. E. Pitcher: *A simpler proof of a theorem of Morse.*

Morse, Transactions of this Society, vol. 27 (1925), and Morse and Van Schaack, Annals of Mathematics, vol. 35 (1934), obtain a complete set of relations between the numbers of critical points of various types and the connectivities (mod 2) of the region in which the function is defined. The function is required to be of class C^2, to have only nondegenerate critical points, and to satisfy simple boundary conditions. A principal intermediate theorem, proved completely only in the first paper, is the following: If the function f has exactly one critical point at the level c and k is its type, and if e is a sufficiently small positive number, then the differences ΔR_i in the connectivities of the sets $f \leqq c-e$ and $f \leqq c+e$ are all 0 except that $\Delta R_k = 1$ or $\Delta R_{k-1} = -1$. The two cases are mutually exclusive. The proof given by Morse is greatly simplified by the author through the use of the Mayer-Vietoris formulas, which appeared several years after the first paper by Morse. Formulas equivalent to the Mayer-Vietoris formulas were used by A. B. Brown in a study of isolated critical points, possibly degenerate, of an analytic function. (Received October 28, 1938.)

455. E. D. Rainville: *Linear differential invariants related to the Laplace integral transformation.*

Certain linear differential equations are left invariant when subjected to the Laplace integral transformation. Such invariance depends essentially on two properties of the Laplace operator. This suggests the introduction of a linear operator σ which has the above mentioned properties in common with the Laplace operator and which is defined with reference to certain linear differential expressions. Let a linear differential form be a finite expression linear and homogeneous in a function and its derivatives, with coefficients which are polynomials in the independent variable. We study the properties of σ when applied to such forms. In four ways all forms which are invariant or pseudo-invariant under σ are classified. The general operational equation in σ with constant coefficients is solved in terms of linear differential invariant forms. Some of the results are extended to the case of forms in two independent variables subjected to iterated σ transformations. A generalization of σ, in which the generalized operator is applied to expressions of which linear differential forms are a very special case, is studied with particular reference to invariant expressions. (Received October 27, 1938.)

456. E. H. Rothe: *Asymptotic solution of a boundary value problem.*

Let y be the solution of the boundary value problem $y''+\mu(y'-y)=\mu f(x)$, $y(0)=y(1)=0$. It is shown that for $\mu\to+\infty$, y approaches that solution of $z'-z=f(x)$ which is zero for $x=1$. In the case of $y''-\mu(y'+y)=\mu f(x)$, y approaches that solution of $-z'-z=f(x)$ which is zero for $x=0$. (Received October 28, 1938.)

457. Leonard Bristow: *Expansion of functions in solutions of functional equations.*

The functional operator $L(x, \lambda)$ is assumed to have the following property: $L(x, \lambda)\to x^p=x^pf(x, \lambda, p)=x^p\sum_\mu f_\mu(p, \lambda)x^\mu$, the series converging suitably for all values of p and the parameter λ. Conditions upon the function $f(x, \lambda, p)$ are found so that there exists a set of values $\{\lambda_m\}$, $(m=0, 1, 2, \cdots)$, such that the equation $L(x, \lambda)\to y(x)=0$ has a set of solutions $\{y_m(x)\}$, $y_m(x)=x^m\sum_s\alpha_{s,m}x^s$, which can be used to expand an arbitrary analytic function $f(x)$; that is, $f(x)=\sum_m y_m(x)$. The expansion converges and represents the function in a circle about the origin. Special cases include: (a) linear differential equations with a regular singular point, (b) Volterra homogeneous integral equations with a regular singularity, (c) linear q-difference equations. (Received September 28, 1938.)

458. Jesse Douglas: *Minimal surfaces of higher topological structure.* II.

This is a continuation, with more details, of the author's papers on the same subject recently presented to the Society; see this Bulletin, vol. 44 (1938), pp. 487–488. (Received October 3, 1938.)

459. Karl Menger: *On non-euclidean and affine geometry.*

The whole geometry of Bolyai and Lobatchewski, including the chapters on parallelism, order, congruence, and convergence, can be based on assumptions concerning the operations of joining and intersecting. The whole affine geometry can be obtained by introducing ideal elements into the non-euclidean geometry. (The results of this paper are published in the Proceedings of the National Academy of Science, October, 1938.) (Received October 3, 1938.)

460. G. H. Peebles: *Some generalizations of the theory of orthogonal polynomials.*

If real constants a, b, c, d, and e are chosen within certain ranges, and if $\rho(x)$ is a non-identically vanishing solution of the differential equation $\rho'(x)/\rho(x)=N(x)/D(x)$, $N(x)=ax+b$, $D(x)=cx^2+dx+e$, the polynomials $q_0(x)=1$, $q_n(x)=(1/\rho)d^n(\rho D^n)/dx^n$, $(n=1, 2, 3, \cdots)$, are an orthogonal set with respect to $\rho(x)$ over an interval dependent on c, d, and e. For values outside these ranges, the set of polynomials loses the property of orthogonality. The purpose of the first part of this paper is to show that the non-orthogonal sets possess properties ordinarily associated with orthogonal sets, namely, each set satisfies the general recursion formula of orthogonal polynomials, each set of a restricted class satisfies the Christoffel-Darboux identity, and each set of a further restricted class has the property of representing suitable functions by means of convergent series. The second part gives a method of constructing orthogonal polynomials for a class of weight functions which change sign in the interval of orthogonality. The method used leads to theorems on the vanishing of determi-

nants formed from the polynomials orthogonal with respect to weight functions which are positive in the interval of orthogonality. (Received October 21, 1938.)

461. M. S. Robertson: *On certain power series having infinitely many zero coefficients.*

Let $f(z)=z+\sum_2^\infty a_n z^n$ be regular for $|z|<1$, and within this circle take on real values for, and only for, real values of z. The author shows that if $a_n=0$ for $n\equiv 0 \pmod p$, where p is any odd integer greater than one, then the coefficients a_n are uniformly bounded: $|a_n|\leqq p-1$. This theorem is not true when p is an even integer. For $p=3$ the sharp inequality $|a_n|\leqq$ max. on $(0\leqq\phi\leqq(\pi/3))$ of $|\cos n\phi/\cos\phi|<2$ is obtained. In this inequality the equal signs are seen to hold for the function $(z-z^2\sec\phi+z^3)\cdot\{1-2z\cos(\pi/3+\phi)+z^2\}^{-1}\cdot\{1-2z\cos(\pi/3-\phi)+z^2\}^{-1}$, where ϕ is chosen as that value for which $|\cos n\phi/\cos\phi|$ attains its maximum over $(0, \pi/3)$. This extremal function varies with n. The rate of growth of $f(z)$ is given by $|f(re^{i\theta})|\leqq 4r/(3^{1/2}(1-r^2))$. These results apply a fortiori to functions $f(z)$ univalent within the unit circle and real on the real axis. (Received September 29, 1938.)

462. S. M. Ulam: *On the distribution of a general measure in any complete metric separable space.*

Let $\mu(A)$ be a completely additive measure defined for a class of subsets of a complete metric separable space E such that open subsets are measurable. There exist compact subsets of E having positive measure and a subset representable as the sum of countably many compact sets having a measure equal to that of the space E. (Received October 4, 1938.)

463. V. W. Adkisson: *Closed disconnected sets with unique maps in the plane.*

An E-set is a closed bounded and disconnected plane point set, each component of which is locally connected, not more than a finite number being of diameter greater than any positive number. The following theorem is proved: An E-set M has a unique map in the plane if and only if one of the following sets of conditions holds: (1) each component of M is an arc or a point, and no interior point of an arc t is a limit point of $M-t$ (referred to as an M.K. set); (2) one component N of M is a triod, and $M-N$ plus the three end points of N is an M.K. set; (3) one component N of M contains a closed 2-cell C and $N-C$ consists of at most a countable number of arcs $a_1, a_2, \cdots$, such that $\bar{a}_i\cdot\bar{a}_j=0$, $(i\neq j)$, and each $\bar{a}_i\cdot C$ is a single point which lies on the boundary of C; each component of $M-N$ is an arc or point such that no interior point of an arc t (of $M-N$) is a limit point of $M-t$; the points of $(\overline{M-N})\cdot N$ are non-cut points of N. (Received October 24, 1938.)

464. Marguerite D. Darkow: *Generalized center-circles.*

The method used by Morley to obtain his chain of theorems on center-circles is capable of generalization and yields infinitely many such chains. Associated with a set of n mutually intersecting lines, there are center-circles of order $1, 2, \cdots, n-2$. Concerning these, the following theorems hold: (1) The ith order center-circle of an n-line passes through the centers of the n ith order center-circles of the n sets of $(n-1)$ lines contained in the given n-line. (2) The n ith order center-circles of the n sets of $(n-1)$ lines formed from a given n-line meet in a point which, when the ith

order center-circle of the n-line is not a point circle, is the node or focus of the limaçon (or the center of a circle, that is, a degenerate limaçon) to which the n given circles are tangent. In particular, when $i=(n-2)/2$, this point of intersection lies on the ith order center-circle of the n-line. (Received October 18, 1938.)

465. Jesse Douglas: *The analytic prolongation of a minimal surface over a rectilinear segment of its boundary.*

Schwarz proved the theorem that if a minimal surface contains a straight line in its interior, then this straight line must be an axis of symmetry of the surface. All proofs so far given of this theorem make essential use of the *interior* position of the line with respect to the minimal surface. For certain interesting applications, however, it is important to have stronger information. Suppose the straight line segment l is part of the *boundary* of a portion M of a minimal surface. Let M be rotated about l through 180°, producing M'. Is then M' the analytic prolongation of M across l? The present paper proves the affirmative answer. An application is then given to the construction of a minimal surface bounded by two given interlacing circles; thus a general theorem is illustrated concerning interlacing contours given by the author in the Journal of Mathematics and Physics, vol. 10 (1931), p. 351. This paper will appear in the Duke Mathematical Journal. (Received October 27, 1938.)

466. M. H. Heins: *Interpolation to functions harmonic and positive or bounded in the unit circle.*

Let P_h denote the class of functions $u(z)$ harmonic and positive for $|z|<1$. Suppose $|z_j|<1$, $(j=1, 2)$, and let $D(z_1, z_2)$ denote the hyperbolic distance between z_1 and z_2. Then $u(z_2)e^{-2D(z_1,z_2)} \leqq u(z_1) \leqq u(z_2)e^{2D(z_1,z_2)}$ for $u \subset P_h$. This inequality enables one to study the general problem of the existence of a function $u(z) \subset P_h$ for which z_k, $(|z_k|<1)$, and $u(z_k)=u_k>0$ are preassigned, $(k=1, 2, \cdots, n$ or $k=1, 2, \cdots)$. The inequalities (1) $u_j e^{-2D(z_j,z_k)} \leqq u_k \leqq u_j e^{2D(z_j,z_k)}$, $(j \neq k;\ j, k=1, 2, \cdots, n$ or $j, k=1, 2, \cdots)$, constitute a necessary condition for the existence of such a $u(z)$. Conversely, if (1) is satisfied, either (1°) for some j^*, k^* one of the weak inequalities (1) is an equality and there exists a unique function $u(z) \subset P_h$ for which $u(z_{j^*})=u_{j^*}$, $u(z_{k^*})=u_{k^*}$, and a necessary and sufficient condition that the problem be compatible is that the remaining u_k be equal to $u(z_k)$; or (2°) all the inequalities (1) are satisfied strongly and there exists a function $u(z) \subset P_h$ with the desired interpolation properties, and all such functions can be determined. The interpolation problem for the class of functions $u(z)$ harmonic for $|z|<1$, $-1<u(z)<1$, $|z|<1$, can be treated similarly. (Received October 31, 1938.)

467. F. B. Jones: *Concerning simple linear Moore spaces and simple continuous curves.*

It is first established that if the author's Axiom 5_1^* (Transactions of this Society, vol. 42 (1937), p. 54) holds true in a Moore space, the space is locally connected, if connected. A simple linear space is one in which the following axiom holds true: If P is a point of a region R, there exists in R a domain D containing P such that D has at most two boundary points. The main result of the paper is that every nondegenerate component of a simple linear Moore space is a simple continuous curve. (A Moore space is a space in which Axiom 0 and parts 1, 2, and 3 of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* hold true.) (Received October 25, 1938.)

468. F. B. Jones: *Concerning the boundary of a complementary domain of a continuous curve.*

Suppose that a space satisfies Axioms 0–4 of R. L. Moore's *Foundations of Point Set Theory*. Among others, the following results are established. I. If K is a locally compact continuum lying in the boundary of a connected domain D, then in order that K be a continuous curve it is necessary and sufficient that K be a subset of a continuous curve which contains no point of D. II. If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every non-degenerate component of the outer boundary of D with respect to E is a simple continuous curve. III. Under the hypothesis of II, the outer boundary of D with respect to E is either acyclic or a simple closed curve. IV. If K is the boundary of a complementary domain D of a locally compact continuous curve M, β is the outer boundary of D with respect to a point E of $S-\overline{D}$, and H is a component of $K-\beta$, then $\overline{H}$ is a continuous curve having only one point in β. (Received October 25, 1938.)

469. J. L. Kelley: *Fixed sets under homeomorphisms.*

The following generalization of a theorem by W. L. Ayres (*Some generalizations of the Scherer fixed point theorem*, Fundamenta Mathematicae, vol. 24 (1925), pp. 125–130) concerning homeomorphisms of locally connected continua into themselves is proved: If T is a homeomorphism carrying a compact continuum M into a subset of itself, there exists a continuum $H \subset M$ without cut points of itself and with $T(H) = H$. If M is decomposed into continua E_0 maximal (see G. T. Whyburn, *Concerning maximal sets*, this Bulletin, vol. 40 (1934), pp. 159–160; also, *Cyclic elements of higher order*, American Journal of Mathematics, vol. 56 (1934), pp. 133–146) with respect to the property of containing no cut points, we may state this theorem: If every homeomorphism T carrying any set $E_0 \subset M$ into a subset of itself leaves a fixed point, then every homeomorphism of M into itself leaves a fixed point; that is, the fixed-point property for homeomorphisms is E_0 extensible. (Received October 21, 1938.)

470. Walter Leighton: *New convergence theorems for continued fractions.*

This paper provides an infinite sequence of convergence theorems, independent of earlier known theorems, for continued fractions of the form (1): $1+K_1^{\infty}(a_n/1)$, where the a_n are complex quantities not equal to zero. The results are combined into the following principal theorem: If the denominators B_r of the first k approximants of the continued fraction (1) are not zero, if the numbers $B_{k-1,\lambda}$ are not zero, if $\left|a_{nk+s+1}B_{k-2,nk+s+1}\right| \leq m < 1$, for each value of s except those congruent to $r_0 \pmod{k}$, and if $|B_rB_{k+r}| \geq |B_r| + |a_1a_2 \cdots a_{r+1}B_{k-1,r+1}|$, $|B_{2k-1,\lambda}| \geq |B_{k-1,\lambda}| + |a_{\lambda+1}a_{\lambda+2} \cdots a_{\lambda+k}B_{k-1,r+k}|$, $(r=0, 1, 2, \cdots, k-1;\ \lambda=1, 2, 3, \cdots)$, the continued fraction converges. The number r_0 is any fixed integer. The notation is that of Perron. Each value of $k=2, 3, 4, \cdots$ yields a theorem for each value of $r_0=0, 1, 2, \cdots, k-1$. (Received October 31, 1938.)

471. Norman Levinson: *General gap Tauberian theorems.* II.

Continuing the problem begun in the Proceedings of the London Mathematical Society, it is shown that the hypothesis for the general higher indices theorem given there is essentially the best possible. It turns out that such kernels as $K(x)=e^{-x^{1/2}}$ require larger gaps than the higher indices type in order that an analogous theorem be

true. For example, instead of $\lambda_{n+1}-\lambda_n \geq D>0$ as in the general higher indices theorem, for $e^{-z^2/2}$ there is needed $\sum 1/\lambda_n<\infty$. The general result (stated very roughly) is that if $\lim_{x\to\infty}\sum_0^\infty a_n\int_{\lambda_n-x}^\infty K(y)dy=s$, if the Fourier transform of $K(x)$ is $k(u)$, if $\Lambda(x)$ is the number of $\lambda_n<x$, if $\alpha(u)$ is monotone increasing and $|k'(u)/k(u)| \leqq \alpha(u)$, and if $\int_1^\infty du/u^2\int_1^{\alpha(u)}\Lambda(x)dx/x<\infty$, then $\sum a_n=s$. Although it is known that there is no unrestricted Tauberian theorem for $e^{-z^2/2}$ if $\{\lambda_n\}$ satisfies nothing more than $\lambda_{n+1}-\lambda_n \geqq D>0$, there is a very mildly restricted result in this case which, stated roughly, requires that $a_n=O(e^{n^2/2-cn\log n})$. Here again there is a general theorem with the same range of application as the theorem stated above. Examples can be given to show that both general theorems referred to here are the best possible. (Received October 27, 1938.)

472. Norman Levinson: *On functions of minimal exponential type.*

Pólya once stated the following theorem for which many proofs have been given: Let $F(z)$ be an entire function of minimal exponential type; if $F(\pm n)$ is uniformly bounded, then $F(z)$ is a constant. The author proved (American Journal of Mathematics) that it suffices if $F(\lambda_n)=O(1)$, $(-\infty<n<\infty)$, if $|\lambda_n-n|<M$. This result is now extended to the theorem: The function $F(z)$ is constant if $F(\lambda_n)=O(1)$ and $|\lambda_n-n|<|n|/(\log|n|)^{1+\epsilon}$, $(-\infty<n<\infty)$. (This last inequality can be made more precise by stating it as a test depending on the convergence of an integral.) What is quite astounding is that this result is the best possible, in the sense that an example can be given of an entire function of minimal type bounded for $z=\lambda_n$ where $|\lambda_n-n|<|n|/\log|n|$, $(-\infty<n<\infty)$, which is not a constant. This result deviates very much from existing density theorems in that it goes as far as it does and yet it need not be true for a set $\{\lambda_n\}$ having a density $D>0$. (Received October 27, 1938.)

473. O. F. G. Schilling: *A remark on normal extensions.*

The author investigates the theory of normal extensions K over function fields of one variable k. The groups G to be considered are groups of order l^m, l a prime. It turns out that, except for special cases, one can reestablish Riemann's existence theorem by purely algebraic methods. The case that l is equal to the characteristic of the underlying field k requires a different treatment. It is found that by prescribing a single branch point, all groups G of order l^m can be realized as Galois groups. (Received October 13, 1938.)

474. A. R. Schweitzer: *On the construction of "configurational" sets of ordered dyads in the foundations of geometry.*

Analogous to his construction of a complete set of open linear chains of dyads in Chapter 2 of his theory of geometrical relations (American Journal of Mathematics, vol. 31), the author constructs complete systems of sets of ordered dyads in n elements ($n=1, 2, 3, \cdots$) on the basis of the self-conjugate dyad $\alpha\alpha$ by means of two principles of generation: (1) successive adjunction of self-conjugate dyads, and (2) successive replacement of any dyad by connected dyads of which it is the resultant. These sets of dyads are placed in two categories: (1) primarily with, and (2) without reference to order. Sets in the second category are abstractly identical with substitutions, and their group properties are derived entirely within the frame of development of Chapter 2 of the author's theory cited. Graphically, the latter sets of dyads are represented by "constellations" or "configurations," that is, combinations of point digons ($\alpha\alpha$), oriented digons ($\alpha\beta$, $\beta\alpha$), oriented trigons ($\alpha\gamma$, $\gamma\beta$, $\beta\alpha$), and so on. Transi-

tion to the corresponding representation of abstract groups is made. The author refers to Chapter 3 of his theory (ibid., vol. 31), to mathematico-psychological theories of "form" of A. B. Kempe and K. Koffka and articles by A. Cayley (ibid., vols. 1, 11), E. H. Moore (ibid., vol. 18), M. Dehn (Mathematische Annalen, vol. 71), and the *Habilitationsschrift* of W. Threlfall (1932). (Received October 24, 1938.)

475. W. J. Trjitzinsky: *Some general developments in the theory of functions of a complex variable.*

The present paper (to appear in Acta Mathematica), and another by the author (*Théorie des fonctions d'une variable complexe définies sur des ensembles généraux*, Annales Scientifiques de l'École Normale, 1938, pp. 119–191), contain an extensive investigation of various classes of functions of a complex variable $f(z)$, each possessing either a unique derivative over some measurable set E or essentially equivalent properties. Also functions $f(z)$ are considered which are limits of sequences of analytic functions $f_\nu(z)$, where each $f_\nu(z)$ is defined over $O_\nu \supset E$. The theory includes as special subcases the theories of analytic, quasi-analytic, and other significant classes of functions. It is important that their representation involves, essentially, integrals over two-dimensional sets of $(\zeta - z)^{-1} d\mu$ where $\zeta = \xi + i\eta$ is the variable of integration and μ is an additive function of Lebesgue measurable sets (absolutely continuous or not). This affords an effective analytic study of the functions in question. Various degrees of "rarefication" of μ in the vicinity of a suitable set $H \subset E$ will secure various preassigned regularity properties; for example, representation as limits of rational functions, differentiability, assigned "degree" of continuity, and unique determination by values on an arc, or on a set of linear positive measure, or by values of derivatives of all orders at a point or on a nondense set, and so on. (Received October 27, 1938.)

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